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Research Article

A FIXED POINT THEOREM IN A COMPLETE METRIC SPACE

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A B S T R A C T

In this paper we prove a common fixed point theorem using reciprocal continuous mapping and compatible mapping of type (A).

INTRODUCTION

The Polish mathematician Stefan Banach [1922] proved a theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. It is well known as a Banach fixed point theorem. This result provides a technique for solving variety of applied problems in mathematical science and engineering. The existence of a fixed point is therefore of paramount importance in several area of mathematics, physics and chemistry. Jungck et al [2] in 1993 generalized the concept of self mappings to commutative self mappings. He further introduced the compatible self mappings of type (A). Pant [3] in 1999 introduced and studied the concept of reciprocal continuous self mappings. In the present paper we shell establish a more generalized common fixed point theorem using the reciprocally continuous mappings and compatible mappings of type (A).

Preliminaries

The following notions have been used to prove the main theorem.

Definition 2.1: Two self mappings f and g on a metric space (X, d) are said to be commute [2] if fg = gf.
Definition 2.2: Two self mappings f and g on a metric space (X, d) are said to be compatible of type (A) [2] if

$\lim_{n\to\infty} d(f g_{x_n}, g g_{x_n}) = 0$ and $\lim_{n\to\infty} d(f g_{x_n}, f f_{x_n}) = 0$.

Whenever $<x_n>$ is a sequence in X such that, $\lim_{n\to\infty} f_{x_n} = \lim_{n\to\infty} g_{x_n} = t$ for some $t \in X$.

Definition 2.3: Two self mappings f and g on a metric space (X, d) are said to be reciprocally continuous [3] if $\lim_{n\to\infty} f g_{x_n} = f_t$ and $\lim_{n\to\infty} g f_{x_n} = g_t$. Whenever $<x_n>$ is a sequence in X such that, $\lim_{n\to\infty} f_{x_n} = \lim_{n\to\infty} g_{x_n} = t$ for some $t \in X$.

Definition 2.4: If S and T are two self mappings of a metric space (X, d) satisfying $g(X) \subseteq f(X)$ then the sequence $\{x_n\}$ in X is called an associated sequence [5] of x_o relative to two self mappings f and g if $g_{x_{2n}} = f_{x_{2n+1}}$ $\forall$ $n \geq 0$.
Throughout this paper X represents a metric space over a metric d.

Comman Fixed Point Theorem

Before proving the main theorem we first prove the following Theorem.

Theorem 3.1: Let f and g are two self mappings of a metric space X satisfying

1. $g(X) \subseteq f(X)$.
2. $d(g_x, g_y) \le k \max.\{ d(f_x, f_y), d(f_x, g_y), d(f_y, g_x), d(f_y, g_y), 1/2\, [d(f_x, g_y) + d(f_y, g_x)]\}$, where $k \in [0, 1)$ $\forall$ x, y $\in$ X.
3. X is complete metric space.

Then the associated sequence $\{x_n\}$ given by $< g_{x_0}, g_{x_1}, g_{x_2}, \ldots\ldots >$ is convergent to some $z \in X$.

Proof: Suppose f and g are two self mappings of a metric space X and let $x_0 \in X$ and $\{x_n\}$ be an associated sequence of x_0.

Since $g_{x_{2n}} = f_{x_{2n+1}}$ and $g_{x_{2n+1}} = f_{x_{2n+2}}$ $\forall$ $n \ge 0$ so,

$d(g_{x_{2n}}, g_{x_{2n+1}}) \le k \max.\{d(f_{x_{2n}}, f_{x_{2n+1}}), d(f_{x_{2n}}, g_{x_{2n+1}}), d(f_{x_{2n+1}}, g_{x_{2n}}), d(f_{x_{2n+1}}, g_{x_{2n+1}}), 1/2\, [d(f_{x_{2n}}, g_{x_{2n+1}}) + d(f_{x_{2n+1}}, g_{x_{2n}})]\}$.

$\le k \max.\{d(f_{x_{2n}}, g_{x_{2n}}), d(f_{x_{2n}}, g_{x_{2n+1}}), d(g_{x_{2n}}, g_{x_{2n}}), d(g_{x_{2n}}, g_{x_{2n+1}}), 1/2\, [d(f_{x_{2n}}, g_{x_{2n+1}}) + d(g_{x_{2n}}, g_{x_{2n}})]\}$.

Since $1/2\, d(g_{x_{2n-1}}, g_{x_{2n+1}}) \le k \max.\{d(g_{x_{2n-1}}, g_{x_{2n}}), d(g_{x_{2n}}, g_{x_{2n+1}})\}$ so,

$d(g_{x_{2n}}, g_{x_{2n+1}}) \le k \max.\{d(g_{x_{2n-1}}, g_{x_{2n}}), d(g_{x_{2n}}, g_{x_{2n+1}}), 1/2\, [d(g_{x_{2n-1}}, g_{x_{2n+1}})\}$.

Again k < 1 so we have, $d(g_{x_{2n}}, g_{x_{2n+1}}) \le k\, d(g_{x_{2n-1}}, g_{x_{2n}})$ (1)

Similarly we have, $d(g_{x_{2n-1}}, g_{x_{2n}}) \le k\, d(g_{x_{2n-2}}, g_{x_{2n-1}})$ (2)

These gives $d(g_{x_{2n}}, g_{x_{2n+1}}) \le k^2\, d(g_{x_{2n-1}}, g_{x_{2n-2}})$ (3)

Continuing this process we get $d(g_{x_{2n}}, g_{x_{2n+1}}) \le k^{2n}\, d(g_{x_0}, g_{x_1})$ (4)

Since k < 1 so $k^{2n} \to 0$ as $n \to \infty$, equation (4) shows that the sequence $\{ g_{x_n} \}$ is a Cauchy sequence in X. But X is complete so it converges to a point $z \in X$.

In a similar way we can prove that $\{ f_{x_n} \}$ is converges to the same point $z \in X$.

Now we prove our main theorem.

Theorem 3.2: Let f and g are two self mappings of a metric space X satisfying

1. $g(X) \subseteq f(X)$.
2. $d(g_x, g_y) \le k \max\{ d(f_x, f_y), d(f_x, g_y), d(f_y, g_x), d(f_y, g_y), 1/2\, [d(f_x, g_y) + d(f_y, g_x)]\}$, where $k \in [0, 1)$ $\forall$ x, y $\in$ X.
3. For any $x_0 \in X$ the associated sequence $\{x_n\}$ for x_o given by $< g_{x_0}, g_{x_1}, g_{x_2}, \ldots\ldots >$ is convergent to some point $z \in X$.
4. The pair (f, g) is reciprocally continuous and compatible.

Then f and g have a unique common fixed point $z \in X$.

Proof: For an associated sequence $\{x_n\}$ of X at x_o we have $g_{x_0}, f_{x_1}, g_{x_{2n}}, f_{x_{2n+1}}, g_{x_{2n+1}}$ ……converges to z as $n \to \infty$ i.e. $\{ g_{x_{2n}} \}$ and $\{ f_{x_{2n+1}} \}$ tends to z.

Since (f, g) is reciprocally continuous, $f_{x_{2n+1}} \to z$, $g_{x_{2n+1}} \to z$ as $n \to \infty$.

So $f g_{x_{2n+1}} \to fz$ and $g f_{x_{2n+1}} \to gz$ as $n \to \infty$ (1)

Again (f, g) is compatible so $\lim_{n\to\infty} d(f g_{x_n}, g g_{x_n}) = 0$ and $\lim_{n\to\infty} d(f g_{x_n}, f f_{x_n}) = 0$.

These gives $f g_{x_{2n+1}} = g g_{x_{2n+1}}$ and $g f_{x_{2n+1}} = f f_{x_{2n+1}}$ (2)

By (1) and (2) we get $fg_{x_{2n+1}} = gg_{x_{2n+1}}$ = fz and $gf_{x_{2n+1}} = ff_{x_{2n+1}}$ = gz.

Now $d(gf_{x_{2n+1}}, g_{x_{2n+1}}) \leq k$ max.{$d(ff_{x_{2n+1}}, f_{x_{2n+1}})$, $d(ff_{x_{2n+1}}, gf_{x_{2n}})$, $d(f_{x_{2n+1}}, g_{x_{2n+1}})$, $d(f_{x_{2n+1}}, gf_{x_{2n+1}})$, $1/2$ [$d(ff_{x_{2n+1}}, g_{x_{2n+1}}) + d(f_{x_{2n+1}}, ff_{x_{2n+1}})$]}.

Letting $n \to \infty$ we get d(gz, z) $\leq$ k max. {d(gz, z), d(gz, z), d(z, z), d(z, gz), $1/2$ [d(gz, z), d(z, gz)]}.

i.e. d(gz, z) $\leq$ k d(gz, z).
Since k < 1 so d(gz, z) = 0 i.e. gz = z (3)

Again $d(gf_{x_{2n+1}}, gg_{x_{2n+1}}) \leq k$ max.{$d(ff_{x_{2n+1}}, fg_{x_{2n+1}})$, $d(ff_{x_{2n+1}}, gg_{x_{2n+1}})$, $d(fg_{x_{2n+1}}, gg_{x_{2n+1}})$, $d(fg_{x_{2n+1}}, gf_{x_{2n+1}})$, $1/2$ [$d(ff_{x_{2n+1}}, gg_{x_{2n+1}}) + d(fg_{x_{2n+1}}, gf_{x_{2n+1}})$]}.

Letting $n \to \infty$ we get

d(gz, fz) $\leq$ k max. {d(gz, fz), d(gz, fz), d(fz, fz), d(fz, gz), $1/2$ [d(gz, fz), d(fz, gz)]}.

i.e. d(gz, fz) $\leq$ k d(gz, fz).

Since k < 1 so d(gz, fz) = 0 i.e. gz = fz (4)

By (3) and (4) we have gz = fz = z (5)

Thus z is a common fixed point of f and g.
Now we prove the uniqueness of z.
For suppose y be another common fixed point of f and g then fy = gy = y (6)

Then d(z, y) = d(gz, gy) $\leq$ k max. {d(fz, fy), d(fz, gy), d(fy, gy), d(fy, gz), $1/2$ [d(fz, gy), d(fy, gz)]}.

Using (5) and (6) we get,

d(fz, gy) $\leq$ k max. {d(z, y), d(z, y), d(y, y), d(y, z), $1/2$ [d(z, y), d(y, z)]}.

i.e. d(z, y) $\leq$ k d(z, y). Since k < 1, we have d(z, y) = 0 i.e. z = y.
Thus z is the unique common fixed point of f and g.

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