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RESEARCH ARTICLE

AN EXTENDED FIXED POINT THEOREM IN THE RATIONAL FORM IN A COMPLETE METRIC SPACE

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ABSTRACT

The aim of this paper is to obtain a fixed point theorem in the form of rational expression. In this paper, we extended the work of Yadav [6] and then show that the results of Banach [1], Kannon[2], Reich[3], Chatterjee[4], Fisher[5] and Yadav[6] are special cases of our theorem.

INTRODUCTION

In 1922, the Polish mathematician Stefan Banach proved a theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. His result is called Banachs fixed point theorem. This result provides a technique for solving variety of applied problems in mathematical science and engineering. In the wide range of mathematical problems the existence of a solution is equivalent to the existence of a fixed point for a suitable map. The existence of a fixed point is therefore of paramount importance in several area of mathematics and physics. Many authors like Kannon [2], Reich [3], Chatterjee [4], Fisher [5] and Yadav [6] have extended, generalized and improved Banach fixed point theorem in different ways. The aim of this paper is to obtain a fixed point theorem in the form of rational expression.

Throughout this paper the complete metric space (X, d) is denoted by X.

Theorem: Let f be a continuous self mapping defined on a complete metric space X with

$$d(f_x, f_y) \le k_1 \frac{d(x,fx)d(y,fy) + d(x,fy)d(y,fx)}{d(x,fx) + d(y,fy) + d(x,fy) + d(y,fx)} + s$$

$$k_2 \frac{d(x,fx)d(y,fx) + d(y,fy)d(x,fy)}{d(x,fx) + d(y,fy) + d(x,fy) + d(y,fx)} + k_3 \frac{d(x,fx)d(x,fx) + d(x,fy)d(y,fx)}{d(x,fx) + d(x,fx) + d(x,fy) + d(y,fx)} +$$

$$k_4 \frac{d(x,fx)d(x,fy) + d(y,fx)d(y,fy)}{d(x,fx) + d(y,fy) + d(x,fy) + d(y,fx)} + k_5 [d(x, fx) + d(y, fy)] + k_6 [d(x, fy) + d(y, fx)] \quad + k_7 d(x, y).$$

$\forall$ x, y $\in$ X, x $\neq$ y and for $k_1, k_2, k_3, k_4, k_5, k_6, k_7 \in [0, 1)$ with $k_1 + k_2 + k_3 + k_4 + 4k_5 + 4k_6 + 2k_7 < 2$, then f has a unique fixed point.

Proof: Let $x_o \in X$ and define a sequence $\{x_n\}$ in X such that $f^n_{x_0} = x_{n+1} \ \forall\, n \in I^+$.

Thus we set $fx_n = x_{n+1}$.

Now $d(x_{n+1},x_n)=d(fx_n, fx_{n-1}) \leq k_1 \frac{d(x_n,f_{x_n})d(x_{n-1},f_{x_{n-1}})+d(x_n,f_{x_{n-1}})d(x_{n-1},f_{x_n})}{d(x_n,f_{x_n})+d(x_{n-1},f_{x_{n-1}})+d(x_n,f_{x_{n-1}})+d(x_{n-1},f_{x_n})}$ + k_2 $\frac{d(x_n,f_{x_n})d(x_{n-1},f_{x_n})+d(x_{n-1},f_{x_{n-1}})d(x_n,f_{x_{n-1}})}{d(x_n,f_{x_n})+d(x_{n-1},f_{x_{n-1}})+d(x_n,f_{x_{n-1}})+d(x_{n-1},f_{x_n})}$ +

$k_3 \frac{d(x_n,f_{x_n})d(x_n,f_{x_n})+d(x_n,f_{x_{n-1}})d(x_{n-1},f_{x_n})}{d(x_n,f_{xn})+d(x_n,f_{xn})+d(x_n,f_{xn-1})+d(x_{n-1},f_{xn})}$ +

$k_4 \frac{d(x_n,f_{x_n})d(x_n,f_{x_{n-1}})+d(x_{n-1},f_{x_n})d(x_{n-1},f_{x_{n-1}})}{d(x_n,f_{x_n})+d(x_{n-1},f_{x_{n-1}})+d(x_n,f_{x_{n-1}})+d(x_{n-1},f_{x_n})}$ + $k_5 [d(x_n, fx_n) + d(x_{n-1}, fx_{n-1})] + k_6 [d(x_n, fx_{n-1}) + d(x_{n-1}, fx_n)] + k_7\, d(x_n, x_{n-1})$.

Or $d(x_{n+1}, x_n) \leq k_1 \frac{d(x_n,x_{n+1})d(x_{n-1},x_n)+d(x_n,x_n)d(x_{n-1},x_{n+1})}{d(x_n,x_{n+1})+d(x_{n-1},x_n)+d(x_n,x_n)+d(x_{n-1},x_{n+1})}$ +

$k_2 \frac{d(x_n,x_{n+1})d(x_{n-1},x_{n+1})+d(x_{n-1},x_n)d(x_n,x_n)}{d(x_n,x_{n+1})+d(x_{n-1},x_n)+d(x_n,x_n)+d(x_{n-1},x_{n+1})}$ + s

$k_3 \frac{d(x_n,x_{n+1})d(x_n,x_{n+1})+d(x_n,x_n)d(x_{n-1},x_{n+1})}{d(x_n,x_{n+1})+d(x_n,x_{n+1})+d(x_n,x_n)+d(x_{n-1},x_{n+1})}$ +

$k_4 \frac{d(x_n,x_{n+1})d(x_n,x_n)+d(x_{n-1},x_{n+1})d(x_{n-1},x_n)}{d(x_n,x_{n+1})+d(x_{n-1},x_n)+d(x_n,x_n)+d(x_{n-1},x_{n+1})}$ + $k_5[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)]$ + $k_6[d(x_n, x_n) + d(x_{n-1}, x_{n+1})] + k_7\, d(x_n, x_{n-1})$.

i.e. $d(x_{n+1}, x_n) \leq k_1 \frac{d(x_n,x_{n+1})d(x_{n-1},x_n)}{d(x_{n-1},x_n)+d(x_{n-1},x_n)} + k_2 \frac{d(x_n,x_{n+1})d(x_{n-1},x_{n+1})}{d(x_{n-1},x_{n+1})+d(x_{n-1},x_{n+1})}$ +

$k_3 \frac{d(x_n,x_{n+1})d(x_n,x_{n+1})}{d(x_n,x_{n+1})+d(x_n,x_{n+1})} + k_4 \frac{d(x_{n-1},x_{n+1})d(x_{n-1},x_n)}{d(x_{n-1},x_{n+1})+d(x_{n-1},x_n)}$ + $k_5 [d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + k_6\, d(x_{n-1}, x_{n+1}) + k_7\, d(x_n, x_{n-1})$.

$\leq \frac{k_1}{2} d(x_n , x_{n+1}) + \frac{k_2}{2} d(x_n , x_{n+1}) + \frac{k_3}{2} d(x_n , x_{n+1}) + \frac{k_4}{2} d(x_{n-1} , x_n) + k_5 [d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + k_6\, d(x_{n-1}, x_{n+1}) + k_7\, d(x_n, x_{n-1})$.

i.e. $d(x_{n+1}, x_n) \leq d(x_{n-1} , x_n) [\frac{k_4}{2} + k_5 + k_6 + k_7] + d(x_{n+1}, x_n) [\frac{k_1}{2} + \frac{k_2}{2} + \frac{k_3}{2} + k_5 + k_6]$.

i.e. $d(x_{n+1}, x_n) [1 - (\frac{k_1}{2} + \frac{k_2}{2} + \frac{k_3}{2} + k_5 + k_6)] \leq d(x_{n-1} , x_n) [\frac{k_4}{2} + k_5 + k_6 + k_7]$.

i.e. $d(x_{n+1}, x_n) \leq \frac{\frac{k_4}{2} + k_5 + k_6 + k_7}{1 - \frac{k_1}{2} - \frac{k_2}{2} - \frac{k_3}{2} - k_5 - k_6} d(x_{n-1} , x_n)$.

Continuing this process we get $d(x_{n+1}, x_n) \leq \left(\frac{\frac{k_4}{2} + k_5 + k_6 + k_7}{1 - \frac{k_1}{2} - \frac{k_2}{2} - \frac{k_3}{2} - k_5 - k_6} \right)^n d(x_0 , x_1)$.

$\leq k^n d(x_0, x_1)$.

Where $k = \dfrac{\frac{k_4}{2} + k_5 + k_6 + k_7}{1 - \frac{k_1}{2} - \frac{k_2}{2} - \frac{k_3}{2} - k_5 - k_6}$.

Now for m > n,

$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})$---------$d(x_{m-1}, x_m)$.

$\leq (k^n + k^{n+1}$-----------$k^{m-1}) d(x_0, x_1)$.

Since $k_1 + k_2 + k_3 + k_4 + 4k_5 + 4k_6 + 2k_7 < 2$. So $d(x_n, x_m) \leq \dfrac{k^n}{1-k} d(x_0, x_1)$

i.e. $d(x_n, x_m) \to \propto$ as $n \to \propto$.

So $\{x_n\}$ is a Cauchy sequence in X. Since X is complete so there exist a point $x \in X$ such that $x_n \to x$ as $n \to \propto$.
Again continuity of f gives $f(x) = f(\lim_{n \to \propto} x_n) = \lim_{n \to \propto} fx_n = \lim_{n \to \propto} x_{n+1} = x$.
Thus f(x) = x i. e. x is a fixed point of f in X.
Now we show that x is unique. For suppose y be other fixed point such that T(y) = y,

$$\text{Then } d(x, y) = d(f_x, f_y) \leq k_1 \frac{d(x,f_y)d(y,f_y) + d(x,fy)d(y,f_x)}{d(x,f_x) + d(y,f_y) + d(x,f_y) + d(y,f_x)} +$$

$$k_2 \frac{d(x,f_y)d(y,f_x) + d(y,f_y)d(x,f_y)}{d(x,f_x) + d(y,f_y) + d(x,f_y) + d(y,f_x)} + k_3 \frac{d(x,f_x)d(x,f_x) + d(x,fy)d(y,f_x)}{d(x,f_x) + d(x,f_x) + d(x,f_y) + d(y,f_x)} +$$

$$k_4 \frac{d(x,f_x)d(x,f_y) + d(y,f_x)d(y,f_y)}{d(x,f_x) + d(y,f_y) + d(x,f_y) + d(y,f_x)} + k_5 [d(x, f_x) + d(y, f_y)] + k_6 [d(x, f_y) + d(y, f_x)]$$

$+ k_7 d(x, y)$.

$$\leq k_1 \frac{d(x,x)d(y,y) + d(x,y)d(y,x)}{d(x,x) + d(y,y) + d(x,y) + d(y,x)} +$$

$$k_2 \frac{d(x,x)d(y,x) + d(y,y)d(x,y)}{d(x,x) + d(y,y) + d(x,y) + d(y,x)} + k_3 \frac{d(x,x)d(x,x) + d(x,y)d(y,x)}{d(x,x) + d(y,y) + d(x,y) + d(y,x)} +$$

$$k_4 \frac{d(x,x)d(x,y) + d(y,x)d(y,y)}{d(x,x) + d(y,y) + d(x,y) + d(y,x)} + k_5 [d(x, x) + d(y, y)] + k_6 [d(x, y) + d(y, x)]$$

$+ k_7 d(x, y)$.

$$\leq k_1 \frac{d(x,y)d(y,x)}{2d(y,x)} + k_3 \frac{d(x,y)d(y,x)}{2d(y,x)} + 2k_6 d(x, y) + k_7 d(x, y).$$

Then $d(x, y) \leq \frac{k_1}{2} d(x,y) + \frac{k_3}{2} d(x,y) + 2k_6 d(x, y) + k_7 d(x, y)$.

So $d(x, y) \leq [\frac{k_1}{2} + \frac{k_3}{2} + 2k_6 + k_7] d(x, y)$.

Which is a contradiction, because $k_1 + k_2 + k_3 + k_4 + 4k_5 + 4k_6 + 2k_7 < 2$.
Thus d(x, y) = 0 i. e. x = y. Hence x is the unique fixed point of f.
Remark

1. If $k_1 = k_2 = k_3 = k_4 = k_5 = k_6 = 0$ then the theorem reduce to Banach [1].
2. If $k_1 = k_2 = k_3 = k_4 = k_5 = k_6 = k_7 = 0$ then the theorem reduce to Kkannan [2].
3. If $k_1 = k_2 = k_3 = k_4 = 0$ then the theorem reduce to Reich [3].
4. If $k_1 = k_2 = k_3 = k_4 = k_6 = 0$ then the theorem reduce to Chatterjee [4].
5. If $k_1 = k_2 = k_3 = k_4 = k_5 = 0$ then the theorem reduce to Fisher [5].

6. If $k_3 = 0$ then the theorem reduce to Yadav [6].

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