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RESEARCH ARTICLE

DIFFERENT ROOTS A RANDOM TRIGONOMETRIC POLYNOMIAL

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A B S T R A C T

Let EN(T; Φ' , Φ'') denote the average number of real roots of the random trigonometric polynomial $T = Tn(\theta, \omega) = \sum_{K=1}^{n} a_K(\omega)\cos k\theta$

In the interval (Φ', Φ''). Clearly , T can have at most 2n zeros in the interval (0, 2π). Assuming that $a_k(\omega)$s to be mutually independent identically distributed normal random variables . Dunnage [1] has shown that in the interval $0 \le \theta \le 2\pi$ all save a certain exceptional set of the functions (T_n (θ,ω)) have $\frac{2n}{\sqrt{3}} + O\left(n^{11/13}(\log n)^{3/13}\right)$ number of zeros , when n is large. We consider the same family of trigonometric polynomials and use the Kac_rice formula for the expectation of the number of real roots and obtain that

$$\text{EN (T; 0, 2π)} \sim \frac{2n}{\sqrt{6}} + O(\log n)$$

This result is better than that of Dunnage since our constant is (1/√2) times his constant and our error term is smaller .The proof is based on the convergence of an integral of which an asymptotic estimation is obtained .

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INTRODUCTION

Let N(T ; Φ' , Φ'') be the number of real zeros of trigonometric polynomial

$$T = Tn(\theta, \omega) = \sum_{K=1}^{n} a_K(\omega) b_K \cos k\theta \quad \textbf{(1)}$$

In the interval (Φ' , Φ'') where the coefficients $a_k(\omega)$ are mutually independent random variables identically distributed according to the normal law and $b_k = k^p$ are positive constants and when multiple zeros are counted only once . Let EN (T ; Φ' , Φ'') denote the expectation of N (T ; Φ', Φ''). Obviously , T_n (Φ, ω) can have at most 2n most zeros in the interval (0 , 2π). Dunnage [1] has shown that *in the interval* $0 \le \theta \le 2\pi$ *all save a certain exceptional set of the functions* $T_n(\theta, \omega)$ *have*

$$\frac{2n}{\sqrt{3}} + O\left(n^{11/13}(\log n)^{3/13}\right)$$

Number of zeros when n is large. The measure of the exceptional set does not exceed (logn)$^{-1}$. subsequently ,Das[2] and Qualls [3] have obtained similar results. In this note our purpose is to show that it is possible to obtain a still lower estimate for the expectation of the number of real roots of (1) by using the method of Loggan & shepp [4]. We show that

$$\text{EN (T; 0, 2π)} \sim \frac{2n}{\sqrt{6}} + O(\log n)$$

This result is better than that of Dunnage since our constant is(1 /√2) times his constant and our error term is smaller.

The Approximation for En (T; 0, 2π)

Let L (n) be a positive-valued function of n such that L(n) and n/ L(n) both approach infinity with n . We take ∈ =L (n)/n throughout. Outside a small exceptional set of ω the value of the function $T_n(\theta, \omega)$ has a negligible number of zeros in each of the intervals (0, ∈) ,(π-∈ , π+∈) and (2π-∈ ,2π). By periodicity , of zeros in each of intervals (0, ∈) and (2π-∈ ,2π) is the same as number in (-∈ ,∈). We shall use the following lemma , which is due to Das [2] .

Lemma

The probability that $T_n(\theta, \omega)$ has more than 1 + (2 / log 2)(logn+2n∈) number of zeros in $\omega - \in \le \theta \le \omega + \in$ does not exceed 2 exp(-n∈) .This lemma is due to Das[2], in the special case $D_n = \sum b_n = n$. The expected number of zeros of T in the interval (Φ' , Φ'') is given by the Kac_Rice formula

$$\text{EN (T ; Φ' , Φ'')} = \int_{-\phi'}^{\phi''} d\theta \int_{-\infty}^{\infty} |\eta| p(0, \eta) d\eta \qquad \textbf{(2)}$$

Where the probability density $p(\xi, \eta)$ $T = \xi$ and $T' = \eta$ is given by the Fourier inversion formula

$$p(\xi,\eta)=\frac{1}{(2\Pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(-i\xi y-i\eta z)\phi(y,z)dydz$$

$\phi(y,z)=E\{\exp(iTy+iT'z)\}$ being the characteristic function of the combined variable (T , T'). In our case, we have

$$T=\sum_{K=1}^{n}a_K(\omega)\cos k\theta \qquad T'=\sum_{K=1}^{n}ka_K(\omega)\sin k\theta$$

$$\phi(y,z)=\exp\left\{-\sum_{K=1}^{n}(y\cos k\theta-zk\sin k\theta)^2\right\}$$

$$p(0,\eta)=\frac{1}{(2\Pi)^2}\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}\exp(1-i\eta z)\exp\left\{-\sum_{K=1}^{n}(y\cos k\theta-zk\sin k\theta)^2\right\}dy$$

For ε > 0,

$$\int_{-\infty}^{\infty}|\eta|\exp(-\in|\eta|)p(0,\eta)d\eta=\mathrm{Re}\frac{1}{(2\Pi)^2}\int_{-\infty}^{\infty}|\eta|\exp(-\in|\eta|)d\eta\int_{-\infty}^{\infty}dz$$

$$\int_{-\infty}^{\infty}\exp(-i\eta z)\exp\left\{-\sum_{1}^{n}(y\cos\theta-zk\sin k\theta)^2\right\}dy$$

$$=\mathrm{Re}\frac{1}{2\Pi^2}\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}\left\{\frac{1}{(\in-iz)^2}+\frac{1}{(\in+iz)^2}\right\}\times\exp\left\{-\sum_{1}^{n}(y\cos k\theta-zk\sin k\theta)^2\right\}dy\quad(3)$$

Where Re stands for the real part.

Here , if we allow coskθ , ksinkθ to be arbitrary , that is we take each of them to be constant in k ,then the probability density p(ξ, η) Of ξ=T(θ) = AX and η= T'(θ) =BX , say , degenerates and we get from (3) the following identity , valid for non-zero A and B which can be chosen suitably .

$$0=\mathrm{Re}\frac{1}{2\Pi^2}\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}\left\{\frac{1}{(\in-iz)^2}+\frac{1}{(\in+iz)^2}\right\}\exp\left\{-(Ay-Bz)^2\right\}dy\quad(4)$$

Subtracting (4) from (3) we get

$$\int_{-\infty}^{\infty}|\eta|\exp(-\in|\eta|)p(0,\eta)d\eta=\mathrm{Re}\frac{1}{2\Pi^2}\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}\left\{\frac{1}{(\in-iz)^2}+\frac{1}{(\in+iz)^2}\right\}$$

$$\times\left\{\exp\left\{-\sum_{1}^{n}(y\cos k\theta-zk\sin k\theta)^2\right\}-\exp\left(-(Ay-Bz)^2\right)\right\}dy$$

$$=\mathrm{Re}\frac{1}{\Pi^2}\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}z\left\{\frac{1}{(\in-iz)^2}+\frac{1}{(\in+iz)^2}\right\}\times\left\{\exp(-Gz^2)-\exp(-Hz^2)\right\}du\quad(5)$$

by transforming the integrals putting y = - uz or y = uz and denoting

$$G=\sum_{K=1}^{n}(u\cos k\theta+k\sin k\theta)^2$$

And H= (Au + B) 2

Now using the identity (Logan and shepp[4] , for α =2),

$$\int_{0}^{\infty}\left\{\exp(-Hz^2)-\exp(-Gz^2)\right\}\frac{dz}{z}=\frac{1}{2}\log(G/H)$$

In the limit as ε→0 we obtain from (5) that

$$\int_{-\infty}^{\infty}|\eta|p(0,\eta)d\eta=\frac{1}{2\Pi^2}\int_{-\infty}^{\infty}\log\left\{\frac{\sum_{K=1}^{n}(u\cos k\theta+k\sin k\theta)^2}{(Au+B)^2}\right\}du\qquad\textbf{(6)}$$

Which has been shown in 3 to be a convergent integral. The double integral appearing in (5) is dominated by a decreasing exponential function. So the involved integrals are uniformly convergent on any interval. Since the integral on the right side of (6) converges , we conclude that both the passage to the limit by letting ε→0 and the subsequent change of the order of integration to produce the equation (6) are justified.

Estimation of the Integral of Equation

In this section we obtain an asymptotic estimation for the integral

$$I=\int_{-\infty}^{\infty}\log\left\{\frac{\sum_{K=1}^{n}(u\cos k\theta+k\sin k\theta)^2}{(Au+B)^2}\right\}du$$

Where A and B are fixed non-zero real numbers. This integral exists in general as a principal value i.e.

$$\lim_{R\to\infty}\int_{-R}^{R}\ldots\ , \text{ if }\quad A^2=\sum_{K=1}^{n}\cos^2 k\theta$$

Let $B^2=\sum_{K=1}^{n}k^2\sin^2 k\theta$ And $C^2=\sum_{K=1}^{n}k\cos k\theta\sin k\theta$

As in Das [2] we have for

$$A^2=\frac{1}{2}\{1+O(1/\log n)\}n=\frac{1}{2}Sn$$

$$B^2=\frac{1}{6}\{1+O(1/\log n)\}n^3=\frac{1}{6}Sn^3$$

and $C^2=O(n^2/\log n)=\dfrac{\beta n^2}{\log n}$, (β = constant) ,

Taking L (n) = logn.

We have always by Cauchy's in equality, AB ≥ C^2. In what follows we will assume that AB > C^2. This happens if θ does not take values from the set {0,± π, ±2π,… } . In fact,

$$A^2B^2-2C^4=\frac{S^2n^4}{12}\left\{1-\frac{24\beta^2}{S^2(\log n)}\right\}\cong\frac{S^2n^4}{12}=A^2B^2\qquad\textbf{(7)}$$

So that

$$I=\int_{-\infty}^{\infty}\log\left\{\frac{\sum_{K=1}^{n}(u\cos k\theta+k\sin k\theta)^2}{(Au+B)^2}\right\}du$$

$$=\int_{0}^{\infty}\log\left\{\frac{(A^2u^2+B^2)^2-4u^2C^4}{A^4u^4+B^4-2u^2A^2B^2}\right\}du$$

$$\cong \int_0^\infty \log \left\{ \frac{A^4u^4 + B^4 + 2u^2A^2B^2}{A^4u^4 + B^4 - 2u^2A^2B^2} \right\} du \quad \text{By (7)}$$

$$= I', \text{ say} \qquad (8)$$

$= \int_0^\infty \log \left\{ \frac{1+x}{1-x} \right\} du$, writing $x = (2u^2A^2B^2) / (A^4u^4 + B^4)$

$$= \int_0^\infty \log \left\{ 1 - \frac{4x}{(1+x^2)} \right\}^{-1/2} du$$

$= \frac{1}{2}\int_0^\infty \{-\log(1-z)\} du$, putting $z = 4x / (1+x^2)$.

Now $x \to 0^+$ as $u \to 0$ or ∞. But $x > \varepsilon > 0$, if $\varepsilon A^4 u^4 - 2u^2A^2 B^2 + \varepsilon B^4 < 0$, which occurs for all u in the interval (d1 $\{O(n^2) / \sqrt{\varepsilon}\}$ –d2) ,where d1 , d2 are functions of ε tending to zero as $\varepsilon \to 0$. Thus for all u in the interval $(0,\infty)$ we can safely assume that $\varepsilon = 1 / n$,and $x = \{1 / L(n)\}$, where n is tending to infinity . Thus

$$I' > 2\int_0^\infty \frac{x}{(1+x)^2} du \qquad = 2\int_0^\infty \left\{ 1 - \frac{1}{L(n)+1} \right\}^2 x du$$

$$= 4\left\{ 1 - \frac{1}{L(n)+1} \right\}^2 \int_0^\infty \frac{u^2A^2B^2}{A^4u^4 + B^4} du$$

$$= \frac{4B}{A}\left\{ 1 - \frac{1}{L(n)+1} \right\}^2 \int_0^\infty \frac{v^2}{v^4+1} dv$$

$$= \left\{ 1 - \frac{1}{L(n)+1} \right\}^2 . \frac{2\Pi n}{\sqrt{6}} \qquad (9)$$

Again $\qquad I' < \frac{1}{2}\int_0^\infty \frac{z}{1-z} du$

$$= \frac{1}{2}\int_0^\infty \frac{4x}{(1-x)^2} du$$

$$= 2\int_0^\infty \left\{ 1 - \frac{1}{L(n)-1} \right\}^2 x du$$

$$= \left\{ 1 - \frac{1}{L(n)+1} \right\}^2 . \frac{2\Pi n}{\sqrt{6}} \qquad (10)$$

Now from (9) and (10) $I' \sim \frac{2\Pi n}{\sqrt{6}}$ (11)

And from (8) and (11) $\quad I \sim \frac{2\Pi n}{\sqrt{6}}$ (12)

Evaluation of En (T; Φ', Φ'')

From (2), (6) and (12), we obtain EN (T; Φ', Φ'') = $\frac{(\Phi''-\Phi')n}{\sqrt{6}}$

In view of our choice of A, B and C

EN (T; $\pi+\varepsilon$, $2\pi-\varepsilon$) = EN (T; ε, $\pi-\varepsilon$)

Again, by the lemma, we have

EN (T ; 0 , ε) + EN (T ; $\pi-\varepsilon$, $\pi+\varepsilon$) + EN (T ; $2\pi-\varepsilon$, 2π)

= EN (T ; $\pi+\varepsilon$, $2\pi-\varepsilon$) $\leq$ 2 { 1 + (2 / log 2)(log n + $2n\varepsilon$) }

Now choosing ε = (log n) / n, the desired result follows.

CONCLUSION

The number of real zeros of trigonometric polynomial

$$T_n(\theta, \omega) = \sum_{K=1}^{n} a_K(\omega) \quad k^p \cos k\theta$$

In the interval (Φ' , Φ'') with coefficients are mutually independent random variables identically distributed according to the normal law and and when multiple zeros are counted only once the number of zeros of the above polynomial is

$$\frac{2n}{\sqrt{3}} + O\left(n^{11/13} (\log n)^{3/13} \right)$$

When n is large. The measure of the exceptional set does not exceed (logn)$^{-1}$.

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