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A FIXED POINT APPROACH TO THE STABILITY OF RECIPROCAL QUADRATIC FUNCTIONAL EQUATION IN INTUITIONISTIC FUZZY BANACH ALGEBRAS

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A B S T R A C T

RESEARCH ARTICLE

In this paper, we prove the generalized Ulam-Hyers stability of quadratic reciprocal functional equation $f(x+y)=\dfrac{f(x)f(y)}{f(x)+f(y)+2\sqrt{f(x)f(y)}}$ associated with intuitionistic fuzzy homomorphisms and intuitionistic fuzzy derivations in intuitionistic fuzzy Banach algebras using Radu's fixed point method.

1. Introduction

In 1940, Ulam [31] posed the famous Ulam stability problem. In 1941, Hyers [16] solved the well-known Ulam stability problem for additive mappings subject to the Hyers condition on approximately additive mappings. He gave rise to the stability theory for functional equations. In 1950, Aoki [2] generalized Hyers' theorem for approximately additive functions. In 1978, Th.M. Rassias [25] provided a generalized version of Hyers for approximately linear mappings. In addition, J.M. Rassias [24, 27] generalized the Hyers stability result by introducing two weaker conditions controlled by a product of different powers of norms and a mixed product-sum of powers of norms, respectively. In 2003, V. Radu [23] proposed a new method, successively developed in [11, 12] to obtain the existence of the exact solutions and the error estimations, based on the fixed point alternative.

Intuitionistic fuzzy sets and Intuitionistic fuzzy metric spaces are studied in [7] and [22], respectively. The concept of intuitionistic fuzzy Banach algebra has been introduced by Bivas Dinda, T.K. Samanta and U.K. Bera [14].

In this paper, we prove the generalized Ulam-Hyers stability of quadratic reciprocal functional equation

$$f(x+y)=\frac{f(x)f(y)}{f(x)+f(y)+2\sqrt{f(x)f(y)}} \tag{1.1}$$

associated with intuitionistic fuzzy homomorphisms and intuitionistic fuzzy derivations in intuitionistic fuzzy Banach algebras using Radu's fixed point method.

2. Definitions On Intuitionistic Fuzzy Banach Algebras

Now, we recall the basic definitions and notations in the setting of intuitionistic fuzzy Banach algebra.

Definition 2.1 A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is said to be continuous t-norm if $*$ satisfies the following conditions:

1. $*$ is commutative and associative;
2. $*$ is continuous;
3. $a*1=a$ for all $a\in[0,1]$;
4. $a*b\le c*d$ whenever $a\le c$ and $b\le d$ for all $a,b,c,d\in[0,1]$.

Definition 2.2 A binary operation $\Diamond:[0,1]\times[0,1]\to[0,1]$ is said to be continuous t-conorm if $\Diamond$ satisfies the following conditions:

1. $\Diamond$ is commutative and associative;
2. $\Diamond$ is continuous;
3. $a\Diamond 0=a$ for all $a\in[0,1]$;
4. $a\Diamond b\le c\Diamond d$ whenever $a\le c$ and $b\le d$ for all $a,b,c,d\in[0,1]$.

Definition 2.3 [14] Let $*$ be a continuous t-norm, $\Diamond$ be a continuous $t-$ conorm, and A be an algebra over the field k (= R or C). An intuitionistic fuzzy normed algebra is an object of the form $(A,\mu,\nu,*,\Diamond)$ where μ,ν are fuzzy sets on $V\times R^+$, μ denotes the degree of membership and ν denotes the degree of non-membership satisfying the following conditions for every $x,y\in A$ and $s,t\in R^+$;

- $\mu(x,t)+\nu(x,t)\le 1$,
- $\mu(x,t)>0$,
- $\mu(x,t)=1$, if and only if $x=0$.
- $\mu(\alpha x,t)=\mu\left(x,\dfrac{t}{|\alpha|}\right)$ for each $\alpha\ne 0$,
- $\mu(x,t)*\mu(y,s)\le \mu(x+y,t+s)$,
- max $\{\mu(x,t),\mu(y,s)\}\le \mu(xy,t+s)$,
- $\lim_{t\to\infty}\mu(x,t)=1$ and $\lim_{t\to 0}\mu(x,t)=0$,
- $\nu(x,t)<1$,
- $\nu(x,t)=0$, if and only if $x=0$.
- $\nu(\alpha x,t)=\nu\left(x,\dfrac{t}{|\alpha|}\right)$ for each $\alpha\ne 0$,
- $\nu(x,t)\Diamond\nu(y,s)\ge \nu(x+y,t+s)$,
- max $\{\nu(x,t),\nu(y,s)\}\ge \nu(xy,t+s)$,
- $\lim_{t\to\infty}\nu(x,t)=0$ and $\lim_{t\to 0}\nu(x,t)=1$.

Example 2.4 Let $(A,\|.\|)$ be a intuitionistic fuzzy normed algebra. Let $a*b=ab$ and $a\Diamond b=\min\{a+b,1\}$ for all $a,b\in[0,1]$. For all $x\in A$ and every $t>0$, consider

$$\mu(x,t)=\begin{cases}\dfrac{t}{t+\|x\|} & if \quad t>0;\\ 0 & if \quad t\le 0;\end{cases} \qquad and$$

$$\nu(x,t)=\begin{cases}\dfrac{\|x\|}{t+\|x\|} & if \quad t>0;\\ 0 & if \quad t\le 0.\end{cases}$$

Then $(A,\mu,\nu,*,\Diamond)$ is an intuitionistic fuzzy normed algebra.

Definition 2.5 A sequence $\{x_n\}_n$ in an intuitionistic fuzzy normed algebra $(A,\mu,\nu,*,\Diamond)$ is said to **converge** to $x\in A$ if for given $r>0, t>0, 0<r<1$, there exist an integer $n_0\in N$ such that $\mu(x_n-x,t)>1-r$ and $\mu(x_n-x,t)<r$ for all $n\ge n_0$.

Definition 2.6 In an intuitionistic fuzzy normed algebra $(A,\mu,\nu,*,\Diamond)$, a sequence $\{x_n\}_n$ converges to $x\in A$ if $\lim_{n\to\infty}\mu(x_n-x,t)=1$ and $\lim_{n\to\infty}\nu(x_n-x,t)=0$ for all $t>0$. In this case, we write $x_n\xrightarrow{IF}x \quad as \quad n\to\infty$.

Definition 2.7 A sequence $\{x_n\}_n$ an intuitionistic fuzzy normed algebra $(A,\mu,\nu,*,\Diamond)$ is said to be **Cauchy sequence** if $\lim_{n\to\infty}\mu(x_{n+p}-x_n,t)=1$ and $\lim_{n\to\infty}\nu(x_{n+p}-x_n,t)=0$ for all $t\in R^+$, $p=1,2,\cdots$.

Definition 2.8 An intuitionistic fuzzy normed algebra $(A,\mu,\nu,*,\Diamond)$ is said to be **complete** if every cauchy sequence in A converges to an element of A.

Definition 2.9 A complete intuitionistic fuzzy normed algebra is called intuitionistic fuzzy Banach algebra.

Theorem 2.10 In an intuitionistic fuzzy normed algebra $(A,\mu,\nu,*,\Diamond)$ two sequences $\{x_n\}_n$ and $\{y_n\}_n$ be such that $x_n\to x$ and $y_n\to y$ then $x_ny_n\to xy$.
Hereafter, throughout this section, assume that A is a linear space, (A',μ',ν') is an intuitionistic fuzzy normed algebra and (B,μ,ν) an intuitionistic fuzzy Banach algebra.

Definition 2.11 A C-linear mapping $H:A\to B$ is called a quadratic reciprocal intuitionistic fuzzy Banach homomorphism if $H(xy)=H(x)H(y)$ for all $x,y\in A$.

Definition 2.12 A C-linear mapping $D:A\to A$ is called a quadratic reciprocal intuitionistic fuzzy Banach derivation if $D(xy)=D(x)\dfrac{1}{y^2}+\dfrac{1}{x^2}D(y)$ for all $x,y\in A$.

Here, we present the upcoming result due to Margolis and Diaz [19] for fixed point theory.

Theorem 2.13 [19] Suppose that for a complete generalized metric space (Ω,δ) and a strictly contractive mapping $T:\Omega\to\Omega$ with Lipschitz constant L. Then, for each given $x\in\Omega$, either

$$d(T^nx,T^{n+1}x)=\infty \qquad \forall \qquad n\ge 0,$$

or there exists a natural number n_0 such that

(FP1) $d(T^nx,T^{n+1}x)<\infty$ for all $n\ge n_0$;

(FP2) The sequence (T^nx) is convergent to a fixed to a fixed point y^* of T

(FP3) y^* is the unique fixed point of T in the set $\Delta=\{y\in\Omega: d(T^{n_0}x,y)<\infty\}$;

(FP4) $d(y^*,y)\le\dfrac{1}{1-L}d(y,Ty)$ for all $y\in\Delta$.

3. Intuitionistic Fuzzy Banach Algebra Stability Results

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1.1) connected to intuitionistic fuzzy homomorphisms and intuitionistic fuzzy derivations in intuitionistic fuzzy Banach algebras using Radu's fixed point method.

Theorem 3.1 Let $f:A\to B$ be a mapping for which there exists a function $P:A\times A\to A'$ with the conditions

$$\lim_{n\to\infty}\mu'\left(P\left(\varsigma^n x,\varsigma_i^n y\right),t\right)=1,$$
$$\lim_{n\to\infty}\nu'\left(P\left(\varsigma_i^n x,\varsigma_i^n y\right),t\right)=0$$

(3.1)

for all $x, y \in A$ and all $t > 0$ where

$$\varsigma_i = \begin{cases} \frac{1}{2} & if \quad i=0 \\ 2 & if \quad i=1 \end{cases}$$

(3.2)

and satisfying the functional inequalities

$$\mu\left(f(x+y)-\frac{f(x)f(y)}{f(x)+f(y)+2\sqrt{f(x)f(y)}},t\right) \geq \mu'\left(P(x,y),t\right)$$

$$\left.\begin{aligned} &\mu\left(f(x+y)-\frac{f(x)f(y)}{f(x)+f(y)+2\sqrt{f(x)f(y)}},t\right) \geq \mu'\left(P(x,y),t\right) \\ &\nu\left(f(x+y)-\frac{f(x)f(y)}{f(x)+f(y)+2\sqrt{f(x)f(y)}},t\right) \leq \nu'\left(P(x,y),t\right)\end{aligned}\right\} \quad (3.3)$$

and

$$\left.\begin{aligned} &\mu\left(f(xy)-f(x)f(y),t\right) \geq \mu'\left(P(x,y),t\right), \\ &\nu\left(f(xy)-f(x)f(y),t\right) \leq \nu'\left(P(x,y),t\right)\end{aligned}\right\}$$

(3.4)

for all $x, y \in A$ and all $t > 0$. If there exists $L = L(i)$ such that the function

$$\wp(x) = P\left(\frac{x}{2},\frac{x}{2}\right),$$

(3.5)

has the property

$$\left.\begin{aligned} &\mu'\left(L\varsigma_i^2\wp(\varsigma_i x),t\right)=\mu'\left(\wp(x),t\right) \\ &\nu'\left(L\varsigma_i^2\wp(\varsigma_i x),t\right)=\nu'\left(\wp(x),t\right)\end{aligned}\right\}$$

(3.6)

for all $x \in A$ and all $t > 0$, then there exists a unique quadratic reciprocal homomorphism $H : A \to B$ satisfying the functional equation (1.1) and

$$\left.\begin{aligned} &\mu\left(f(x)-H(x),t\right) \geq \mu'\left(\wp(x),\frac{L^{1-i}}{1-L}t\right) \\ &\nu\left(f(x)-H(x),t\right) \leq \nu'\left(\wp(x),\frac{L^{1-i}}{1-L}t\right)\end{aligned}\right\}$$

(3.7)

for all $x \in A$ and all $t > 0$.

Proof.

Consider the set

$$\Lambda = \{h \mid A \to B,\ h(0)=0\}$$

and introduce the generalized metric on Λ,

$$d(h,f) = \inf\left\{\begin{aligned} &L\in(0,\infty): \\ &\left\{\begin{aligned} &\mu(h(x)-f(x),t) \geq \mu'(\wp(x),Lt), x\in A, t>0 \\ &\nu(h(x)-f(x),t) \leq \nu'(\wp(x),Lt), x\in A, t>0\end{aligned}\right\}\end{aligned}\right\}. \quad (3.8)$$

It is easy to see that (3.8) is complete with respect to the defined metric. Define $J : \Lambda \to \Lambda$ by $Jh(x) = \varsigma_i^2 h(\varsigma_i x)$, for all $x \in A$. Now, from (3.8) and $h, f \in \Lambda$

$$\inf\left\{L\in(0,\infty): \begin{aligned} &\left\{\begin{aligned} &\mu(h(x)-f(x),t) \\ &\geq \mu'(\wp(x),t), x\in A, t>0\} \\ &\mu(\varsigma_i^2 h(\varsigma_i x)-\varsigma_i^2 f(\varsigma_i x),t) \\ &\geq \mu'\left(\wp(\varsigma_i x),\frac{t}{\varsigma_i^2}\right), x\in A, t>0\} \\ &\mu(\varsigma_i^2 h(\varsigma_i x)-\varsigma_i^2 f(\varsigma_i x),t) \\ &\quad \geq \mu'(\wp(x),Lt), x\in A, t>0\} \\ &\mu(Jh(x)-Jf(x),t) \\ &\quad \geq \mu'(\wp(x),Lt), x\in A, t>0\} \end{aligned}\right. \\ &\left\{\begin{aligned} &\nu(h(x)-f(x),t) \\ &\leq \nu'(\wp(x),t), x\in A, t>0\} \\ &\nu(\varsigma_i^2 h(\varsigma_i x)-\varsigma_i^2 f(\varsigma_i x),t) \\ &\leq \nu'\left(\wp(\varsigma_i x),\frac{t}{\varsigma_i^2}\right), x\in A, t>0\} \\ &\nu(\varsigma_i^2 h(\varsigma_i x)-\varsigma_i^2 f(\varsigma_i x),t) \\ &\quad \leq \nu'(\wp(x),Lt), x\in A, t>0\} \\ &\nu(Jh(x)-Jf(x),t) \\ &\quad \leq \nu'(\wp(x),Lt), x\in A, t>0\} \end{aligned}\right. \end{aligned}\right\}$$

This implies J is a strictly contractive mapping on Λ with Lipschitz constant L.

Replacing (x, y) by (x, x) in (3.3), we reach

$$\inf\left\{\begin{array}{l}1\in(0,\infty):\\ \left\{\begin{array}{l}\mu(f(2x)-\frac{f(x)}{4},t)\geq\mu'\left(P(x,x),t\right)\\ \nu(f(2x)-\frac{f(x)}{4},t)\leq\nu'\left(P(x,x),t\right)\end{array}\right.\end{array}\right\}$$

(3.9)

for all $x\in A$ and all $t>0$. Now, from (3.9) and (3.6) for the case $i=0$, we reach

$$\inf\left\{L^{1-0}\in(0,\infty):\right\}\left\{\begin{array}{l}\mu(f(2x)-\frac{f(x)}{4},t)\\ \qquad\geq\mu'\left(P(x,x),t\right)\\ \mu(4f(2x)-f(x),t)\\ \qquad\geq\mu'\left(P(x,x),\frac{t}{4}\right)\\ \mu(Jf(x)-f(x),t)\\ \qquad\geq\mu'\left(\wp(x),Lt\right)\end{array}\right.$$

$$\inf\left\{L^{1-0}\in(0,\infty):\right\}\left\{\begin{array}{l}\nu(f(2x)-\frac{f(x)}{4},t)\\ \qquad\leq\nu'\left(P(x,x),t\right)\\ \nu(4f(2x)-f(x),t)\\ \qquad\leq\nu'\left(P(x,x),\frac{t}{4}\right)\\ \nu(Jf(x)-f(x),t)\\ \qquad\leq\nu'\left(\wp(x),Lt\right)\end{array}\right.$$

(3.10)

for all $x\in A$ and all $t>0$. Again by interchanging x into $\frac{x}{2}$ in (3.9) and using (3.6) for the case $i=1$, we get

$$\inf\left\{\begin{array}{l}L^{1-1}\in(0,\infty)\\ \left\{\begin{array}{l}\mu(f(x)-\frac{1}{4}f\left(\frac{x}{2}\right),t)\geq\mu'\left(P\left(\frac{x}{2},\frac{x}{2}\right),t\right)\\ \mu(f(x)-Jf(x),t)\geq\mu'\left(\wp(x),t\right)\end{array}\right.\\ :\\ \left\{\begin{array}{l}\nu(f(x)-\frac{1}{4}f\left(\frac{x}{2}\right),t)\leq\nu'\left(P\left(\frac{x}{2},\frac{x}{2}\right),t\right)\\ \nu(f(x)-Jf(x),t)\leq\nu'\left(\wp(x),t\right)\end{array}\right.\end{array}\right\}$$

(3.11)

for all $x\in A$ and all $t>0$. Thus, from (3.10) and (3.11), we arrive

$$\inf\left\{\begin{array}{l}L^{1-i}\in(0,\infty):\\ \left\{\begin{array}{l}\mu(f(x)-Jf(x),t)\geq\mu'(\wp(x),L^{1-i}t),\\ \nu(f(x)-Jf(x),t)\leq\nu'(\wp(x),L^{1-i}t)\end{array}\right.\end{array}\right\}$$

(3.12)

for all $x\in A$ and all $t>0$. Hence property (FP1) holds.

By (FP2), it follows that there exists a fixed point H of J in Λ such that

$$\lim_{n\to\infty}\mu\left(\varsigma_i^{2n}f(\varsigma_i^n x)-H(x),t\right)=1,$$
$$\lim_{n\to\infty}\nu\left(\varsigma_i^{2n}f(\varsigma_i^n x)-H(x),t\right)=0$$

for all $x\in A$ and all $t>0$.

In order to prove $H:A\to B$ is quadratic reciprocal, replacing (x,y) by $\left(\varsigma_i^n x,\varsigma_i^n y\right)$ and multiplying by ς_i^{2n} in (3) and using the definition of $H(x)$, and then letting $n\to\infty$, we see that H satisfies (1) for all $x,y\in A$ and all $t>0$. So it follows that

$$\left.\begin{array}{l}\mu(H(xy)-H(x)H(y),t)\\ =\mu(2^{4n}\left(H(2^n x.2^n y)-H(2^n x)H(2^n y)\right),2^{4n}t)\\ \geq\mu'(P(2^n x,2^n y),t),\\ \nu(H(xy)-H(x)H(y),t)\\ =\nu(2^{4n}\left(H(2^n x.2^n y)-H(2^n x)H(2^n y)\right),2^{4n}t)\\ \leq\nu'(P(2^n x,2^n y),t)\end{array}\right\}$$

for all $x,y\in A$ and all $t>0$. Letting $n\to\infty$ in above inequalities, we obtain

$$\left.\begin{array}{l}\mu(H(xy)-H(x)H(y),t)=1,\\ \nu(H(xy)-H(x)H(y),t)=0\end{array}\right\}$$

for all $x,y\in A$ and all $t>0$. Hence H is a quadratic reciprocal homomorphism.

By (FP3), H is the unique fixed point of J in the set $\Delta=\{H\in\Lambda:d(f,H)<\infty\}, H$ is the unique function such that

$$\mu(f(x)-H(x),t)\geq\mu'(\wp(x),L^{1-i}t),$$
$$\nu(f(x)-H(x),t)\leq\nu'(\wp(x),L^{1-i}t)$$

for all $x\in A$ and all $t>0$. Finally by (FP4), we obtain

$$\mu\left(f(x)-H(x),t\right)\geq\mu'\left(\wp(x),\frac{L^{1-i}}{1-L}t\right),$$

$$\nu\left(f(x)-H(x),t\right)\leq\nu'\left(\wp(x),\frac{L^{1-i}}{1-L}t\right)$$

for all $x\in A$ and all $t>0$. So, the proof is complete.

The following corollary is an immediate consequence of Theorem 3.1 which shows that (1.1) can be stable.

Corollary 3.2 Suppose that a function $f : A \to B$ satisfies the double inequality

$$\mu\left(f(x+y) - \frac{f(x)f(y)}{f(x)+f(y)+2\sqrt{f(x)f(y)}}, t \right) \geq \begin{cases} \mu'(\lambda, t), & \\ \mu'\left(\lambda\left(\|x\|^{\ell} + \|y\|^{\ell}\right), t\right), & \ell \neq -2 \\ \mu'\left(\lambda \|x\|^{\ell} \|y\|^{\ell}, t\right), & \ell \neq -1 \\ \mu'\left(\lambda\left\{\|x\|^{\ell}\|y\|^{\ell} + \left(\|x\|^{2\ell} + \|y\|^{2\ell}\right)\right\}, t\right), & \ell \neq -1 \end{cases}$$

$$\nu\left(f(x+y) - \frac{f(x)f(y)}{f(x)+f(y)+2\sqrt{f(x)f(y)}}, t \right) \leq \begin{cases} \nu'(\lambda, t), & \\ \nu'\left(\lambda\left(\|x\|^{\ell} + \|y\|^{\ell}\right), t\right), & \ell \neq -2 \\ \nu'\left(\lambda \|x\|^{\ell} \|y\|^{\ell}, t\right), & \ell \neq -1 \\ \nu'\left(\lambda\left\{\|x\|^{\ell}\|y\|^{\ell} + \left(\|x\|^{2\ell} + \|y\|^{2\ell}\right)\right\}, t\right), & \ell \neq -1 \end{cases}$$

(3.13)

and

$$\mu\left(H(xy) - H(x)H(y), t\right) \geq \begin{cases} \mu'(\lambda, t), \\ \mu'\left(\lambda\left(\|x\|^{\ell} + \|y\|^{\ell}\right), t\right), \\ \mu'\left(\lambda \|x\|^{\ell} \|y\|^{\ell}, t\right), \\ \mu'\left(\lambda\left\{\|x\|^{\ell}\|y\|^{\ell} + \left(\|x\|^{2\ell} + \|y\|^{2\ell}\right)\right\}, t\right), \end{cases}$$

$$\nu\left(H(xy) - H(x)H(y), t\right) \leq \begin{cases} \nu'(\lambda, t), \\ \nu'\left(\lambda\left(\|x\|^{\ell} + \|y\|^{\ell}\right), t\right), \\ \nu'\left(\lambda \|x\|^{\ell} \|y\|^{\ell}, t\right), \\ \nu'\left(\lambda\left\{\|x\|^{\ell}\|y\|^{\ell} + \left(\|x\|^{2\ell} + \|y\|^{2\ell}\right)\right\}, t\right), \end{cases}$$

for all $x, y \in A$ and all $t > 0$, where λ, ℓ are constants with $\lambda > 0$. Then there exists a unique quadratic reciprocal homomorphism $H : A \to B$ such that the double inequality

$$\mu\left(f(x) - H(x), t\right) \geq \begin{cases} \mu'\left(|3|\lambda, t\right) \\ \mu'\left(2\,|2^2 - 2^{-\ell}|\,\lambda \|x\|^{\ell}, 2^{\ell} t\right) \\ \mu'\left(|2^2 - 2^{-2\ell}|\,\lambda \|x\|^{2\ell}, 2^{2\ell} t\right) \\ \mu'\left(3\,|2^2 - 2^{-2\ell}|\,\lambda \|x\|^{2\ell}, 2^{2\ell} t\right) \end{cases}$$

$$\nu\left(f(x) - H(x), t\right) \leq \begin{cases} \nu'\left(|3|\lambda, t\right) \\ \nu'\left(2\,|2^2 - 2^{-\ell}|\,\lambda \|x\|^{\ell}, 2^{\ell} t\right) \\ \nu'\left(|2^2 - 2^{-2\ell}|\,\lambda \|x\|^{2\ell}, 2^{2\ell} t\right) \\ \nu'\left(3\,|2^2 - 2^{-2\ell}|\,\lambda \|x\|^{2\ell}, 2^{2\ell} t\right) \end{cases}$$ holds for all $x \in A$ and all $t > 0$.

Proof. Set

$$\mu'\left(P(\varsigma_i^n x, \varsigma_i^n y), \frac{t}{\varsigma_i^{2n}}\right) = \begin{cases} \mu'\left(\lambda, \varsigma_i^{-2n} t\right), \\ \mu'\left(\lambda\left(\|x\|^{\ell} + \|y\|^{\ell}\right), \varsigma_i^{(-2-\ell)n} t\right), \\ \mu'\left(\lambda \|x\|^{\ell} \|y\|^{\ell}, \varsigma_i^{(-2-2\ell)n} t\right), \\ \mu'\left(\lambda\left\{\|x\|^{\ell}\|y\|^{\ell} + \left(\|x\|^{2\ell} + \|y\|^{2\ell}\right)\right\}, \varsigma_i^{(-2-2\ell)n} t\right), \end{cases} = \begin{cases} \to 1 \; as \;\; k \to \infty \\ \to 1 \; as \;\; k \to \infty \\ \to 1 \; as \;\; k \to \infty \\ \to 1 \; as \;\; k \to \infty \end{cases}$$

$$\nu'\left(P(\varsigma_i^n x, \varsigma_i^n y), \frac{t}{\varsigma_i^{2n}}\right) = \begin{cases} \nu'\left(\lambda, \varsigma_i^{-2n} t\right), \\ \nu'\left(\lambda\left(\|x\|^{\ell} + \|y\|^{\ell}\right), \varsigma_i^{(-2-\ell)n} t\right), \\ \nu'\left(\lambda \|x\|^{\ell} \|y\|^{\ell}, \varsigma_i^{(-2-2\ell)n} t\right), \\ \nu'\left(\lambda\left\{\|x\|^{\ell}\|y\|^{\ell} + \left(\|x\|^{2\ell} + \|y\|^{2\ell}\right)\right\}, \varsigma_i^{(-2-2\ell)n} t\right), \end{cases} = \begin{cases} \to 0 \; as \;\; k \to \infty \\ \to 0 \; as \;\; k \to \infty \\ \to 0 \; as \;\; k \to \infty \\ \to 0 \; as \;\; k \to \infty \end{cases}$$

for all $x \in A$ and all $t > 0$. Thus, the relation (3.1) holds. It follows from (3.5), (3.6) and (3.13), we get

$$\mu'\left(P\left(\frac{x}{2},\frac{x}{2}\right),t\right)=\begin{cases}\mu'(\lambda,t),\\ \mu'\left(2\lambda\|x\|^{\ell},2^{\ell}t\right),\\ \mu'\left(\lambda\|x\|^{2\ell},2^{2\ell}t\right),\\ \mu'\left(3\lambda\|x\|^{2\ell},2^{2\ell}t\right),\end{cases}$$

$$\nu'\left(P\left(\frac{x}{2},\frac{x}{2}\right),t\right)=\begin{cases}\nu'(\lambda,t),\\ \nu'\left(2\lambda\|x\|^{\ell},2^{\ell}t\right),\\ \nu'\left(\lambda\|x\|^{2\ell},2^{2\ell}t\right),\\ \nu'\left(3\lambda\|x\|^{2\ell},2^{2\ell}t\right)\end{cases}$$

for all $x,y\in A$ and all $t>0$. Also from (3.6), we have

$$\mu'\left(\varsigma_i^2\wp(\varsigma_i x),t\right)=\begin{cases}\mu'\left(\lambda,\varsigma_i^{-2}t\right),\\ \mu'\left(2\lambda\|x\|^{\ell},\varsigma_i^{-2-\ell}t\right),\\ \mu'\left(\lambda\|x\|^{2\ell},\varsigma_i^{-2-2\ell}t\right),\\ \mu'\left(3\lambda\|x\|^{2\ell},\varsigma_i^{-2-2\ell}t\right),\end{cases}$$

$$\nu'\left(\varsigma_i^2\wp(\varsigma_i x),t\right)=\begin{cases}\nu'\left(\lambda,\varsigma_i^{-2}t\right),\\ \nu'\left(2\lambda\|x\|^{\ell},\varsigma_i^{-2-\ell}t\right),\\ \nu'\left(\lambda\|x\|^{2\ell},\varsigma_i^{-2-2\ell}t\right),\\ \nu'\left(3\lambda\|x\|^{2\ell},\varsigma_i^{-2-2\ell}t\right)\end{cases}$$

for all $x\in A$ and all $t>0$. Hence, the inequality (3.7) is true for

	L	$i=0$	L	$i=1$
(1)	2^{-2}	0	2^{2}	0
(2)	$2^{-2-\ell}$	$\ell<2$	$2^{2+\ell}$	$\ell>2$
(3)	$2^{-2-2\ell}$	$\ell<-1$	$2^{2+2\ell}$	$\ell>-1$
(4)	$2^{-2-2\ell}$	$\ell<-1$	$2^{2+2\ell}$	$\ell>-1.$

$$\left.\begin{aligned}\mu\left(f(x)-H(x),t\right)&\geq\mu'\left(\wp(x),\frac{2^{-2}}{1-2^{-2}}t\right)\\&=\mu'\left(\lambda,\frac{t}{3}\right)\\ \nu\left(f(x)-H(x),t\right)&\leq\nu'\left(\wp(x),\frac{2^{-2}}{1-2^{-2}}t\right)\\&=\mu'\left(\lambda,\frac{t}{3}\right)\end{aligned}\right\}$$

for all $x\in A$ and all $t>0$. Also, for condition (1) and $i=1$, we get

$$\left.\begin{aligned}\mu\left(f(x)-H(x),t\right)&\geq\mu'\left(\wp(x),\frac{t}{1-2^{2}}\right)=\mu'\left(\lambda,\frac{t}{-3}\right)\\ \nu\left(f(x)-H(x),t\right)&\leq\nu'\left(\wp(x),\frac{t}{1-2^{2}}\right)=\nu'\left(\lambda,\frac{t}{-3}\right)\end{aligned}\right\}$$

for all $x\in A$ and all $t>0$. Again, for condition (2) and $i=0$, we obtain

$$\left.\begin{aligned}\mu\left(f(x)-H(x),t\right)&\geq\mu'\left(\wp(x),\frac{2^{-2-\ell}}{1-2^{-2-\ell}}t\right)\\&=\mu'\left(2\lambda\|x\|^{\ell},\frac{2^{\ell}t}{\left(2^{2}-2^{-\ell}\right)}\right)\\ \nu\left(f(x)-H(x),t\right)&\leq\nu'\left(\wp(x),\frac{2^{-2-\ell}}{1-2^{-2-\ell}}t\right)\\&=\nu'\left(2\lambda\|x\|^{\ell},\frac{2^{\ell}t}{\left(2^{2}-2^{-\ell}\right)}\right)\end{aligned}\right\}$$

for all $x\in A$ and all $t>0$. Also, for condition (2) and $i=1$, we arrive

$$\left.\begin{aligned}\mu\left(f(x)-H(x),t\right)&\geq\mu'\left(\wp(x),\frac{1}{1-2^{2+\ell}}t\right)\\&=\mu'\left(\lambda\|x\|^{\ell},\frac{2^{\ell}t}{2^{-\ell}-2^{2}}\right)\\ \nu\left(f(x)-H(x),t\right)&\leq\nu'\left(\wp(x),\frac{1}{1-2^{2+\ell}}t\right)\\&=\nu'\left(\lambda\|x\|^{\ell},\frac{2^{\ell}t}{2^{-\ell}-2^{2}}\right)\end{aligned}\right\}$$

for all $x\in A$ and all $t>0$. The rest of the proof is similar to that of previous cases. This finishes the proof.

The proof of the following Theorem 3.3 and Corollary 3.4 is similar lines to the Theorem 3.1 and Corollary 3.2.

Theorem 3.3 Let $f:A\to A$ be a mapping for which there exists a function $P:A\times A\to A'$ with the double condition

$$\begin{aligned}&\lim_{n\to\infty}\mu'\left(P\left(\varsigma_i^n x,\varsigma_i^n y\right),t\right)=1,\\&\lim_{n\to\infty}\nu'\left(P\left(\varsigma_i^n x,\varsigma_i^n y\right),t\right)=0\end{aligned}\qquad(3.14)$$

for all $x,y\in A$ and all $t>0$ where

$$\varsigma_i=\begin{cases}\frac{1}{2} & if\quad i=0\\ 2 & if\quad i=1\end{cases}\qquad(3.15)$$

and satisfying the double functional inequality

$$\left.\begin{aligned}&\mu\left(f(x+y)-\frac{f(x)f(y)}{f(x)+f(y)+2\sqrt{f(x)f(y)}},t\right)\\&\qquad\geq\mu'\left(P(x,y),t\right)\\&\nu\left(f(x+y)-\frac{f(x)f(y)}{f(x)+f(y)+2\sqrt{f(x)f(y)}},t\right)\\&\qquad\leq\nu'\left(P(x,y),t\right)\end{aligned}\right\}$$

(3.16)

And

$$\left.\begin{aligned}\mu\left(f(xy)-f(x)\frac{1}{y^2}-\frac{1}{x^2}f(y),t\right)\geq\mu'(P(x,y),t)\\ \nu\left(f(xy)-f(x)\frac{1}{y^2}-\frac{1}{x^2}f(y),t\right)\leq\nu'(P(x,y),t)\end{aligned}\right\}\quad(3.17)$$

for all $x, y \in A$ and all $t > 0$. If there exists $L = L(i)$ such that the function

$$\wp(x) = P\left(\frac{x}{2},\frac{x}{2}\right),\quad(3.18)$$

has the property

$$\begin{aligned}\mu'\left(L\varsigma_i^2\wp(\varsigma_i x),t\right)=\mu'\left(\wp(x),t\right),\\ \nu'\left(L\varsigma_i^2\wp(\varsigma_i x),t\right)=\nu'\left(\wp(x),t\right)\end{aligned}\quad(3.19)$$

for all $x \in A$ and all $t > 0$, then there exists a unique quadratic reciprocal derivation $D : A \to A$ satisfying the functional equation (1.1) and

$$\begin{aligned}\mu\left(f(x)-D(x),t\right)\geq\mu'\left(\wp(x),\frac{L^{1-i}}{1-L}t\right),\\ \nu\left(f(x)-D(x),t\right)\leq\nu'\left(\wp(x),\frac{L^{1-i}}{1-L}t\right)\end{aligned}\quad(3.20)$$

for all $x \in A$ and all $t > 0$.

Corollary 3.4 Suppose that a function $f : A \to A$ satisfies the inequalities (3.13) and

$$\left.\begin{aligned}&\mu\left(D(xy)-D(x)\frac{1}{y^2}-\frac{1}{x^2}D(y),t\right)\\ &\geq\begin{cases}\mu'\left(\lambda,t\right),\\ \mu'\left(\lambda\left(\|x\|^{\ell}+\|y\|^{\ell}\right),t\right),\\ \mu'\left(\lambda\|x\|^{\ell}\|y\|^{\ell},t\right),\\ \mu'\left(\lambda\left\{\|x\|^{\ell}\|y\|^{\ell}+\left(\|x\|^{2\ell}+\|y\|^{2\ell}\right)\right\},t\right),\end{cases}\\ &\nu\left(D(xy)-D(x)\frac{1}{y^2}-\frac{1}{x^2}D(y),t\right)\\ &\leq\begin{cases}\nu'\left(\lambda,t\right),\\ \nu'\left(\lambda\left(\|x\|^{\ell}+\|y\|^{\ell}\right),t\right),\\ \nu'\left(\lambda\|x\|^{\ell}\|y\|^{\ell},t\right),\\ \nu'\left(\lambda\left\{\|x\|^{\ell}\|y\|^{\ell}+\left(\|x\|^{2\ell}+\|y\|^{2\ell}\right)\right\},t\right),\end{cases}\end{aligned}\right\}$$

for all $x, y \in A$ and all $t > 0$, where λ, ℓ are constants with $\lambda > 0$. Then there exists a unique quadratic reciprocal derivation $D : A \to A$ such that the inequalities

$$\left.\begin{aligned}&\mu\left(f(x)-D(x),t\right)\geq\begin{cases}\mu'\left(|3|\lambda,t\right)\\ \mu'\left(2|2^2-2^{-\ell}|\lambda\|x\|^{\ell},2^{\ell}t\right)\\ \mu'\left(|2^2-2^{-2\ell}|\lambda\|x\|^{2\ell},2^{2\ell}t\right)\\ \mu'\left(3|2^2-2^{-2\ell}|\lambda\|x\|^{2\ell},2^{2\ell}t\right)\end{cases}\\ &\nu\left(f(x)-D(x),t\right)\leq\begin{cases}\nu'\left(|3|\lambda,t\right)\\ \nu'\left(2|2^2-2^{-\ell}|\lambda\|x\|^{\ell},2^{\ell}t\right)\\ \nu'\left(|2^2-2^{-2\ell}|\lambda\|x\|^{2\ell},2^{2\ell}t\right)\\ \nu'\left(3|2^2-2^{-2\ell}|\lambda\|x\|^{2\ell},2^{2\ell}t\right)\end{cases}\end{aligned}\right\}$$

holds for all $x \in A$ and all $t > 0$.

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