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ULAM - HYERS STABILITY OF EULER - LAGRANGE QUADRATIC FUNCTIONAL EQUATION IN INTUITIONISTIC FUZZY BANACH SPACES: DIRECT AND FIXED POINT METHODS

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A B S T R A C T

RESEARCH ARTICLE

In this paper, authors verify the generalized Ulam - Hyers stability of the Euler - Lagrange quadratic functional equation in Intuitionistic Fuzzy Banach Spaces using direct and fixed point methods.

1. Introduction

The training of stability problems for functional equations is coupled to a inquiry of S.M. Ulam [28] subject to the stability of group homomorphisms and confirmatory answer specified by D.H. Hyers [12] for Banach spaces. It was further generalized and excellent results obtained by number of authors [2, 11, 21, 22, 23].

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 3, 7, 8, 10, 13, 15, 18].

In this paper, authors verify the generalized Ulam - Hyers stability of the following Euler - Lagrange quadratic functional equation

$$\begin{aligned}(r+s)\,r\,g(x)+(r+s)\,sg(y) \\ = g(rx+sy)+rsg(x-y)\end{aligned} \tag{1.1}$$

where r,s are positive integers with $r,s \neq 0$ in Intuitionistic Fuzzy Banach Spaces using direct and fixed point methods.

2. Definitions On Intuitionistic Fuzzy Banach Space

Now, we recall the basic definitions and notations in the setting of intuitionistic fuzzy normed space.

Definition 2.1 A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is said to be continuous t-norm if $*$ satisfies the following conditions:

1. $*$ is commutative and associative;
2. $*$ is continuous;
3. $a*1=a$ for all $a\in[0,1]$;
4. $a*b\le c*d$ whenever $a\le c$ and $b\le d$ for all $a,b,c,d\in[0,1]$.

Definition 2.2 A binary operation $\Diamond:[0,1]\times[0,1]\to[0,1]$ is said to be continuous t-conorm if $\Diamond$ satisfies the following conditions:

1. $\Diamond$ is commutative and associative;
2. $\Diamond$ is continuous;
3. $a\Diamond 0=a$ for all $a\in[0,1]$;
4. $a\Diamond b\le c\Diamond d$ whenever $a\le c$ and $b\le d$ for all $a,b,c,d\in[0,1]$.

Using the notions of continuous t-norm and t-conorm, Saadati and Park [24] introduced the concept of intuitionistic fuzzy normed space as follows:

Definition 2.3 The five-tuple $(X,\mu,\nu,*,\Diamond)$ is said to be an intuitionistic fuzzy normed space (for short, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\Diamond$ is a continuous $t-$ conorm, and μ,ν are fuzzy sets on $X\times(0,\infty)$ satisfying the following conditions. For every $x,y\in X$ and $s,t>0$

- $\mu(x,t)+\nu(x,t)\le 1$,
- $\mu(x,t)>0$,
- $\mu(x,t)=1$, if and only if $x=0$.
- $\mu(\alpha x,t)=\mu\left(x,\dfrac{t}{|\alpha|}\right)$ for each $\alpha\neq 0$,
- $\mu(x,t)*\mu(y,s)\le\mu(x+y,t+s)$,
- $\mu(x,\cdot):(0,\infty)\to[0,1]$ is continuous,
- $\lim_{t\to\infty}\mu(x,t)=1$ and $\lim_{t\to 0}\mu(x,t)=0$,
- $\nu(x,t)<1$,
- $\nu(x,t)=0$, if and only if $x=0$.
- $\nu(\alpha x,t)=\nu\left(x,\dfrac{t}{|\alpha|}\right)$ for each $\alpha\neq 0$,
- $\nu(x,t)\Diamond\nu(y,s)\ge\nu(x+y,t+s)$,
- $\nu(x,\cdot):(0,\infty)\to[0,1]$ is continuous,
- $\lim_{t\to\infty}\nu(x,t)=0$ and $\lim_{t\to 0}\nu(x,t)=1$.

In this case, (μ, ν) is called an intuitionistic fuzzy norm.

Example 2.4 Let $(X, \|\cdot\|)$ be a normed space. Let $a * b = ab$ and $a \Diamond b = \min\{a+b, 1\}$ for all $a, b \in [0,1]$. For all $x \in X$ and every $t > 0$, consider

$$\mu(x,t) = \begin{cases} \dfrac{t}{t+\|x\|} & if \quad t > 0; \\ 0 & if \quad t \le 0; \end{cases} \qquad and$$

$$\nu(x,t) = \begin{cases} \dfrac{\|x\|}{t+\|x\|} & if \quad t > 0; \\ 0 & if \quad t \le 0. \end{cases}$$

Then $(X, \mu, \nu, *, \Diamond)$ is an IFN-space.

The concepts of convergence and Cauchy sequences in an intuitionistic fuzzy normed space are investigated in [24].

Definition 2.5 Let $(X, \mu, \nu, *, \Diamond)$ be an IFN-space. Then, a sequence $x = \{x_k\}$ is said to be intuitionistic fuzzy convergent to a point $L \in X$ if $\lim_{k\to\infty} \mu(x_k - L, t) = 1 \quad and \quad \lim_{k\to\infty} \nu(x_k - L, t) = 0$ for all $t > 0$. In this case, we write $x_k \xrightarrow{IF} L \quad as \quad k \to \infty$

Definition 2.6 Let $(X, \mu, \nu, *, \Diamond)$ be an IFN-space. Then, $x = \{x_k\}$ is said to be intuitionistic fuzzy Cauchy sequence if $\mu(x_{k+p} - x_k, t) = 1 \quad and \quad \nu(x_{k+p} - x_k, t) = 0$ for all $t > 0$, and $p = 1, 2 \cdots$.

Definition 2.7 Let $(X, \mu, \nu, *, \Diamond)$ be an IFN-space. Then $(X, \mu, \nu, *, \Diamond)$ is said to be complete if every intuitionistic fuzzy Cauchy sequence in $(X, \mu, \nu, *, \Diamond)$ is intuitionistic fuzzy convergent $(X, \mu, \nu, *, \Diamond)$.

Hereafter and subsequently, assume that X is a linear space, (Z, μ', ν') is an intuitionistic fuzzy normed space and (Y, μ, ν) an intuitionistic fuzzy Banach space. Now, we use the following notation for a given mapping $g : X \to Y$ such that

$$Dg_{rx}^{sy} = (r+s)\, r\, g(x) + (r+s)\, s g(y) - g(rx+sy) - rsg(x-y)$$

where r, s are positive integers with $r, s \ne 0$ for all $x, y \in X$.

3. IFNS: Stability Results : Direct Method

In this section, we investigate the generalized Ulam-Hyers stability of the functional equation (1) in INFS using direct method.

Theorem 3.1 Let $\eta \in \{1, -1\}$. Let $\varphi : X \times X \to Z$ be a function such that for some $0 < \left(\dfrac{p}{(r+s)^2}\right)^\eta < 1$,

$$\left.\begin{aligned} \mu'\left(\varphi\left((r+s)^{n\eta} x, (r+s)^{n\eta} y\right), t\right) &\ge \mu'\left(p^{n\eta} \varphi(x,y), t\right) \\ \nu'\left(\varphi\left((r+s)^{n\eta} x, (r+s)^{n\eta} y\right), t\right) &\le \nu'\left(p^{n\eta} \varphi(x,y), t\right) \end{aligned}\right\} \tag{3.1}$$

for all $x \in X$ and all $t > 0$ and

$$\left.\begin{aligned} \lim_{n\to\infty} \mu'\left(\varphi\left((r+s)^{\eta n} x, (r+s)^{\eta n} y\right), (r+s)^{2\eta n} t\right) &= 1 \\ \lim_{n\to\infty} \nu'\left(\varphi\left((r+s)^{\eta n} x, (r+s)^{\eta n} y\right), (r+s)^{2\eta n} t\right) &= 0 \end{aligned}\right\}$$

(3.2)

for all $x, y \in X$ and all $t > 0$. Let $g : X \to Y$ be a function satisfying the inequality

$$\left.\begin{aligned} \mu\left(Dg_{rx}^{sy}(x,y), t\right) &\ge \mu'\left(\varphi(x,y), t\right) \\ \nu\left(Dg_{rx}^{sy}(x,y), t\right) &\le \nu'\left(\varphi(x,y), t\right) \end{aligned}\right\}$$

(3.3)

for all $x, y \in X$ and all $t > 0$. Then there exists a unique quadratic mapping $\mathrm{Q} : X \to Y$ satisfying (1.1) and

$$\left.\begin{aligned} \mu\left(g(x) - \mathrm{Q}(x), t\right) &\ge \mu'\left(\varphi(x,x), |\,(r+s)^2 - p\,|\, t\right) \\ \nu\left(g(x) - \mathrm{Q}(x), t\right) &\le \nu'\left(\varphi(x,x), |\,(r+s)^2 - p\,|\, t\right) \end{aligned}\right\}$$

(3.4)

for all $x \in X$ and all $t > 0$.

Proof. **Case (i):** Let $\eta = 1$.
Setting (x, y) by (x, x) in (3.3), we have

$$\left.\begin{aligned} \mu\left(g\left((r+s)x\right) - (r+s)^2 g(x), t\right) &\ge \mu'\left(\varphi(x,x), t\right) \\ \nu\left(g\left((r+s)x\right) - (r+s)^2 g(x), t\right) &\le \nu'\left(\varphi(x,x), t\right) \end{aligned}\right\}$$

(3.5)

for all $x, y \in X$ and all $t > 0$. It follows from (3.5) and (IFN4), (IFN10), we arrive

$$\left.\begin{aligned} \mu\Big(\frac{g\left((r+s)x\right)}{(r+s)^2} - g(x), \frac{t}{(r+s)^2}\Big) &\ge \mu'\left(\varphi(x,x), t\right) \\ \nu\Big(\frac{g\left((r+s)x\right)}{(r+s)^2} - g(x), \frac{t}{(r+s)^2}\Big) &\le \nu'\left(\varphi(x,x), t\right) \end{aligned}\right\}$$

(3.6)

for all $x \in X$ and all $t > 0$. Substituting x by $(r+s)^n x$ in (3.6), we have

$$\left.\begin{aligned}&\mu\left(\frac{g((r+s)^{n+1}x)}{(r+s)^2}-g((r+s)^n x),\frac{t}{(r+s)^2}\right)\\&\qquad\geq\mu'\left(\varphi((r+s)^n x,(r+s)^n x),t\right)\\&\nu\left(\frac{g((r+s)^{n+1}x)}{(r+s)^2}-g((r+s)^n x),\frac{t}{(r+s)^2}\right)\\&\qquad\leq\nu'\left(\varphi((r+s)^n x,(r+s)^n x),t\right)\end{aligned}\right\}$$

(3.7)

for all $x\in X$ and all $t>0$. It is easy to verify from (3.7) and using (3.1), (IFN4), (IFN10) that

$$\left.\begin{aligned}&\mu\left(\frac{g((r+s)^{n+1}x)}{(r+s)^{(2n+2)}}-\frac{g((r+s)^n x)}{(r+s)^{2n}},\frac{t}{(r+s)^{2n+2}}\right)\\&\qquad\geq\mu'\left(\varphi(x,x),\frac{t}{p^n}\right)\\&\nu\left(\frac{g((r+s)^{n+1}x)}{(r+s)^{(2n+2)}}-\frac{g((r+s)^n x)}{(r+s)^{2n}},\frac{t}{(r+s)^{2n+2}}\right)\\&\qquad\leq\nu'\left(\varphi(x,x),\frac{t}{p^n}\right)\end{aligned}\right\}$$

(3.8)

for all $x\in X$ and all $t>0$. Interchanging t into $p^n t$ in (3.8), we have

$$\left.\begin{aligned}&\mu\left(\frac{g((r+s)^{n+1}x)}{(r+s)^{(2n+2)}}-\frac{g((r+s)^n x)}{(r+s)^{2n}},\frac{t\cdot p^n}{(r+s)^{2n+2}}\right)\\&\qquad\geq\mu'\left(\varphi(x,x),t\right)\\&\nu\left(\frac{g((r+s)^{n+1}x)}{(r+s)^{(2n+2)}}-\frac{g((r+s)^n x)}{(r+s)^{2n}},\frac{t\cdot p^n}{(r+s)^{2n+2}}\right)\\&\qquad\leq\nu'\left(\varphi(x,x),t\right)\end{aligned}\right\}$$

(3.9)

for all $x\in X$ and all $t>0$. It is easy to see that

$$\frac{g((r+s)^n x)}{(r+s)^{2n}}-g(x)=\sum_{i=0}^{n-1}\frac{g((r+s)^{i+1}x)}{(r+s)^{2(i+1)}}-\frac{g((r+s)^i x)}{(r+s)^{2i}}$$

(3.10)

for all $x\in X$. From equations (3.9) and (3.10), we get

$$\left.\begin{aligned}&\mu\left(\frac{g((r+s)^n x)}{(r+s)^{2n}}-g(x),\sum_{i=0}^{n-1}\frac{p^i t}{(r+s)^{2i+2}}\right)\\&=\mu\left(\sum_{i=0}^{n-1}\frac{g((r+s)^{i+1}x)}{(r+s)^{2(i+1)}}-\frac{g((r+s)^i x)}{(r+s)^{2i}},\sum_{i=0}^{n-1}\frac{p^i t}{(r+s)^{2i+2}}\right)\\&\nu\left(\frac{g((r+s)^n x)}{(r+s)^{2n}}-g(x),\sum_{i=0}^{n-1}\frac{p^i t}{(r+s)^{2i+2}}\right)\\&=\nu\left(\sum_{i=0}^{n-1}\frac{g((r+s)^{i+1}x)}{(r+s)^{2(i+1)}}-\frac{g((r+s)^i x)}{(r+s)^{2i}},\sum_{i=0}^{n-1}\frac{p^i t}{(r+s)^{2i+2}}\right)\end{aligned}\right\}(3.11)$$

for all $x\in X$ and all $t>0$. From equations (3.10) and (3.11), we have

$$\left.\begin{aligned}&\mu\left(\frac{g((r+s)^n x)}{(r+s)^{2n}}-g(x),\sum_{i=0}^{n-1}\frac{p^i t}{(r+s)^{2i+2}}\right)\\&\geq\prod_{i=0}^{n-1}\mu\left(\frac{g((r+s)^{i+1}x)}{(r+s)^{2(i+1)}}-\frac{g((r+s)^i x)}{(r+s)^{2i}},\frac{p^i tr}{(r+s)^{2i+2}}\right)\\&\nu\left(\frac{g((r+s)^n x)}{(r+s)^{2n}}-g(x),\sum_{i=0}^{n-1}\frac{p^i t}{(r+s)^{2i+2}}\right)\\&\leq\coprod_{i=0}^{n-1}\nu\left(\frac{g((r+s)^{i+1}x)}{(r+s)^{2(i+1)}}-\frac{g((r+s)^i x)}{(r+s)^{2i}},\frac{p^i t}{(r+s)^{2i+2}}\right)\end{aligned}\right\}$$

(3.12)

where

$$\prod_{i=0}^{n-1}c_j=c_1*c_2*\cdots*c_n \ and \ \coprod_{i=0}^{n-1}d_j=d_1\Diamond d_2\Diamond\cdots\Diamond d_n$$

for all $x\in X$ and all $t>0$. Hence

$$\left.\begin{aligned}&\mu\left(\frac{g((r+s)^n x)}{(r+s)^{2n}}-g(x),\sum_{i=0}^{n-1}\frac{p^i t}{(r+s)^{2i+2}}\right)\\&\geq\prod_{i=0}^{n-1}\mu'\left(\varphi(x,x),t\right)=\mu'\left(\varphi(x,x),t\right)\\&\nu\left(\frac{g((r+s)^n x)}{(r+s)^{2n}}-g(x),\sum_{i=0}^{n-1}\frac{p^i t}{(r+s)^{2i+2}}\right)\\&\leq\coprod_{i=0}^{n-1}\nu'\left(\varphi(x,x),t\right)=\nu'\left(\varphi(x,x),t\right)\end{aligned}\right\}$$

(3.13)

for all $x\in X$ and all $t>0$. Replacing x by $(r+s)^m x$ in (3.13) and using (3.2), (IFN4), (IFN10), we obtain

$$\left.\begin{array}{l}\mu\left(\frac{g((r+s)^{n+m}x)}{(r+s)^{2(n+m)}}-\frac{g((r+s)^{m}x)}{(r+s)^{2m}},\sum_{i=0}^{n-1}\frac{p^{i}\,t}{(r+s)^{(2i+2m+2)}}\right)\\ \geq \mu'\left(\varphi((r+s)^{m}x,(r+s)^{m}x),t\right)=\mu'\left(\varphi(x,x),\frac{t}{p^{m}}\right)\\ \nu\left(\frac{g((r+s)^{n+m}x)}{(r+s)^{2(n+m)}}-\frac{g((r+s)^{m}x)}{(r+s)^{2m}},\sum_{i=0}^{n-1}\frac{p^{i}\,t}{(r+s)^{(2i+2m+2)}}\right)\\ \leq \nu'\left(\varphi((r+s)^{m}x,(r+s)^{m}x),t\right)=\nu'\left(\varphi(x,x),\frac{t}{p^{m}}\right)\end{array}\right\}$$

(3.14)

for all $x\in X$ and all $t>0$ and all $m,n\geq 0$. Replacing t by $p^{m}t$ in (3.14), we get

$$\left.\begin{array}{l}\mu\left(\frac{g((r+s)^{n+m}x)}{(r+s)^{2(n+m)}}-\frac{g((r+s)^{m}x)}{(r+s)^{2m}},\sum_{i=0}^{n-1}\frac{p^{i+m}\,t}{(r+s)^{(2i+2m+2)}}\right)\\ \qquad\geq \mu'\left(\varphi(x.x),t\right)\\ \nu\left(\frac{g((r+s)^{n+m}x)}{(r+s)^{2(n+m)}}-\frac{g((r+s)^{m}x)}{(r+s)^{2m}},\sum_{i=0}^{n-1}\frac{p^{i+m}\,t}{(r+s)^{(2i+2m+2)}}\right)\\ \qquad\leq \nu'\left(\varphi(x,x),t\right)\end{array}\right\}$$

(3.15)

for all $x\in X$ and all $t>0$ and all $m,n\geq 0$. The relation (3.15) implies that

$$\left.\begin{array}{l}\mu\left(\frac{g((r+s)^{n+m}x)}{(r+s)^{2(n+m)}}-\frac{g((r+s)^{m}x)}{(r+s)^{2m}},t\right)\\ \qquad\geq \mu'\left(\varphi(x,x),\frac{t}{\sum_{i=m}^{n-1}\frac{p^{i}}{(r+s)^{2i+2}}}\right)\\ \nu\left(\frac{g((r+s)^{n+m}x)}{(r+s)^{2(n+m)}}-\frac{g((r+s)^{m}x)}{(r+s)^{2m}},t\right)\\ \qquad\leq \nu'\left(\varphi(x,x),\frac{t}{\sum_{i=m}^{n-1}\frac{p^{i}}{(r+s)^{2i+2}}}\right)\end{array}\right\}$$

(3.16)

holds for all $x\in X$ and all $t>0$ and all $m,n\geq 0$. Since $0<p<(r+s)^{2}$ and $\sum_{i=0}^{n}\left(\frac{p}{(r+s)^{2}}\right)^{i}<\infty$. The Cauchy criterion for convergence in IFNS shows that the sequence $\left\{\frac{g((r+s)^{n}x)}{(r+s)^{2n}}\right\}$ is Cauchy in (Y,μ,ν). Since (Y,μ,ν) is a complete IFN-space this sequence converges to some point $\mathsf{Q}(x)\in Y$. So, one can define the mapping $\mathsf{Q}:X\to Y$ by

$$\lim_{n\to\infty}\mu\left(\frac{g((r+s)^{n}x)}{(r+s)^{2n}}-Q(x),t\right)=1,$$

$$\lim_{n\to\infty}\nu\left(\frac{g((r+s)^{n}x)}{(r+s)^{2n}}-Q(x),t\right)=0$$

for all $x\in X$ and all $t>0$. Hence

$$\frac{g((r+s)^{n}x)}{(r+s)^{2n}}\xrightarrow{IF}\mathsf{Q}(x),\quad as\quad n\to\infty.$$

Letting $m=0$ in (3.16), we arrive

$$\left.\begin{array}{l}\mu\left(\frac{g((r+s)^{n}x)}{(r+s)^{2n}}-g(x),t\right)\\ \geq \mu'\left(\varphi(x,x),\frac{t}{\sum_{i=0}^{n-1}\frac{p^{i}}{(r+s)^{2i+2}}}\right)\\ \nu\left(\frac{g((r+s)^{n}x)}{(r+s)^{2n}}-g(x),t\right)\\ \leq \nu'\left(\varphi(x,x),\frac{t}{\sum_{i=0}^{n-1}\frac{p^{i}}{(r+s)^{2i+2}}}\right)\end{array}\right\}$$

(3.17)

for all $x\in X$ and all $t>0$. Letting n tend to infinity in (17), we have

$$\left.\begin{array}{l}\mu(\mathsf{Q}(x)-g(x),t)\geq \mu'\left(\varphi(x,x),t((r+s)^{2}-p)\right)\\ \nu(\mathsf{Q}(x)-g(x),t)\leq \nu'\left(\varphi(x,x),t((r+s)^{2}-p)\right)\end{array}\right\}\tag{3.18}$$

for all $x\in X$ and all $t>0$. To prove Q satisfies (1.1), replacing (x,y) by $((r+s)^{n}x,(r+s)^{n}y)$ in (3.3) respectively, we obtain

$$\left.\begin{array}{l}\mu\left(\frac{1}{(r+s)^{2n}}Dg_{rx}^{sy}((r+s)^{n}x,(r+s)^{n}y),t\right)\\ \geq \mu'\left(\varphi((r+s)^{n}x,(r+s)^{n}y),(r+s)^{2n}t\right)\\ \nu\left(\frac{1}{(r+s)^{2n}}Dg_{rx}^{sy}((r+s)^{n}x,(r+s)^{n}y),t\right)\\ \leq \nu'\left(\varphi((r+s)^{n}x,(r+s)^{n}y),(r+s)^{2n}t\right)\end{array}\right\}$$

(3.19)

for all $x\in X$ and all $t>0$. Now,

$$\mu((r+s)\,r\,Q(x)+(r+s)\,sQ(y)$$
$$\qquad -Q(rx+sy)-rsQ(x-y))$$

$$\geq \mu((r+s)\,r\,Q(x)-\frac{(r+s)r}{(r+s)^{2n}}g(x),\frac{t}{5})$$
$$*\,\mu((r+s)\,sQ(y)-\frac{(r+s)s}{(r+s)^{2n}}g(y),\frac{t}{5})$$
$$*\mu(-Q(rx+sy)+\frac{1}{(r+s)^{2n}}g(rx+sy),\frac{t}{5})$$
$$*\mu(-rsQ(x-y)+\frac{rs}{(r+s)^{2n}}g(x-y),\frac{t}{5})$$
$$*\mu(\frac{(r+s)r}{(r+s)^{2n}}g(x)+\frac{(r+s)s}{(r+s)^{2n}}g(y)$$
$$-\frac{1}{(r+s)^{2n}}g(rx+sy)-\frac{rs}{(r+s)^{2n}}g(x-y),\frac{t}{5}) \qquad (3.20)$$

and

$$\nu((r+s)\,r\,Q(x)+(r+s)\,sQ(y)$$
$$-Q(rx+sy)-rsQ(x-y))$$
$$\geq \nu((r+s)\,r\,Q(x)-\frac{(r+s)r}{(r+s)^{2n}}g(x),\frac{t}{5})$$
$$\lozenge\,\nu((r+s)\,sQ(y)-\frac{(r+s)s}{(r+s)^{2n}}g(y),\frac{t}{5})$$
$$\lozenge\,\nu(-Q(rx+sy)+\frac{1}{(r+s)^{2n}}g(rx+sy),\frac{t}{5})$$
$$\lozenge\,\nu(-rsQ(x-y)+\frac{rs}{(r+s)^{2n}}g(x-y),\frac{t}{5})$$
$$\lozenge\,\nu(\frac{(r+s)r}{(r+s)^{2n}}g(x)+\frac{(r+s)s}{(r+s)^{2n}}g(y)$$
$$-\frac{1}{(r+s)^{2n}}g(rx+sy)-\frac{rs}{(r+s)^{2n}}g(x-y),\frac{t}{5}) \qquad (3.21)$$

for all $x \in X$ and all $t > 0$. Also

$$\left.\begin{array}{l}\lim\limits_{n\to\infty}\mu(\frac{1}{(r+s)^{2n}}Dg_{rx}^{sy}((r+s)^n x,(r+s)^n y),\frac{t}{5})=1\\ \lim\limits_{n\to\infty}\nu(\frac{1}{(r+s)^{2n}}Dg_{rx}^{sy}((r+s)^n x,(r+s)^n y),\frac{t}{5})=0\end{array}\right\}$$

for all $x \in X$ and all $t > 0$. Letting $n \to \infty$ in (3.20), (3.21) and using (3.22), we observe that Q fulfills (1.1). Therefore, Q is a quadratic mapping. In order to prove $Q(x)$ is unique, let $Q'(x)$ be another quadratic functional equation satisfying (1.1) and (3.4). Hence,

$$\mu(Q(x)-Q'(x),t)$$
$$\geq \mu\left(Q((r+s)^n x)-g((r+s)^n x),\frac{t.(r+s)^{2n}}{2}\right)$$
$$*\mu\left(g((r+s)^n x)-Q'((r+s)^n x),\frac{t.(r+s)^{2n}}{2}\right)$$
$$\geq \mu'\left(\varphi((r+s)^n x,(r+s)^n x),\frac{t(r+s)^{2n}}{2}|(r+s)^2-p|\right)$$
$$\geq \mu'\left(\varphi(x,x),\frac{t(r+s)^{2n}|(r+s)^2-p|}{2\cdot p^n}\right)$$
$$\nu(Q(x)-Q'(x),t)$$
$$\leq \nu\left(Q((r+s)^n x)-g((r+s)^n x),\frac{t.(r+s)^{2n}}{2}\right)$$
$$\lozenge\,\nu\left(g((r+s)^n x)-Q'((r+s)^n x),\frac{t.(r+s)^{2n}}{2}\right)$$
$$\leq \nu'\left(\varphi((r+s)^n x,(r+s)^n x),\frac{t(r+s)^{2n}}{2}|(r+s)-p|\right)$$
$$\leq \nu'\left(\varphi(x,x),\frac{t(r+s)^{2n}|(r+s)^2-p|}{2\cdot p^n}\right)$$

for all $x \in X$ and all $t > 0$. Since $\lim\limits_{n\to\infty}\frac{t(r+s)^{2n}|(r+s)^2-p|}{2p^n}=\infty,$ we obtain

$$\left.\begin{array}{l}\lim\limits_{n\to\infty}\mu'\left(\varphi(x),\frac{t(r+s)^{2n}|(r+s)^2-p|}{2\cdot p^n}\right)=1\\ \lim\limits_{n\to\infty}\nu'\left(\varphi(x),\frac{t(r+s)^{2n}|(r+s)^2-p|}{2\cdot p^n}\right)=0\end{array}\right\}$$

for all $x \in X$ and all $t > 0$. Thus

$$\left.\begin{array}{l}\mu(Q(x)-Q'(x),t)=1\\ \nu(Q(x)-Q'(x),t)=0\end{array}\right\} \qquad (3.22)$$

for all $x \in X$ and all $t > 0$. Hence, $Q(x) = Q'(x)$. Therefore, $Q(x)$ is unique.

Case 2: For $\eta = -1$. Putting x by $\frac{x}{(r+s)}$ in (3.5), we get

$$\left.\begin{array}{l}\mu\left(g(x)-(r+s)^2 g\left(\frac{x}{(r+s)}\right),t\right)\\ \geq \mu'\left(\varphi\left(\frac{x}{(r+s)},\frac{x}{(r+s)}\right),t\right)\\ \nu\left(g(x)-(r+s)^2 g\left(\frac{x}{(r+s)}\right),t\right)\\ \leq \nu'\left(\varphi\left(\frac{x}{(r+s)},\frac{x}{(r+s)}\right),t\right)\end{array}\right\}$$

(3.23)

for all $x, y \in X$ and all $t > 0$. The rest of the proof is similar to that of Case 1. This completes the proof of the theorem.

The following corollary is an immediate consequence of Theorem 3.1, regarding the stability of (1.1).

Corollary 3.2 Suppose that a function $g : X \to Y$ satisfies the double inequality

$$\left.\begin{array}{l}\mu\left(Dg_{rx}^{sy}(x,y),t\right)\\ \geq \begin{cases}\mu'(\lambda,t), & \\ \mu'\left(\lambda\left(|x|^a+|y|^b\right),t\right), & a,b\neq 2\\ \mu'\left(\lambda|x|^a|y|^b,t\right), & a+b\neq 2\\ \mu'\left(\lambda\left\{|x|^a|y|^b+\left(|x|^{a+b}+|y|^{a+b}\right)\right\},t\right), & a+b\neq 2\end{cases}\\ \nu\left(Dg_{rx}^{sy}(x,y),t\right)\\ \leq \begin{cases}\nu'(\lambda,t), & \\ \nu'\left(\lambda\left(|x|^a+|y|^b\right),t\right), & a,b\neq 2\\ \nu'\left(\lambda|x|^a|y|^b,t\right), & a+b\neq 2\\ \nu'\left(\lambda\left\{|x|^a|y|^b+\left(|x|^{a+b}+|y|^{a+b}\right)\right\},t\right), & a+b\neq 2\end{cases}\end{array}\right\}$$

(3.24)

for all $x, y \in X$ and all $t > 0$, where λ, a, b are constants with $\lambda > 0$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\left.\begin{array}{l}\mu\left(g(x)-Q(x),t\right)\\ \geq \begin{cases}\mu'\left(\lambda,t\,|(r+s)^2-1|\right),\\ \mu'\left(\left[\lambda|x|^a+\lambda|x|^b\right],t\begin{bmatrix}|(r+s)^2-(r+s)^a|\\ +|(r+s)^2-(r+s)^b|\end{bmatrix}\right),\\ \mu'\left(\lambda|x|^{a+b},t\,|(r+s)^2-(r+s)^{a+b}|\right),\\ \mu'\left(2\lambda|x|^{a+b},t\,|(r+s)^2-(r+s)^{a+b}|\right)\end{cases}\\ \nu\left(g(x)-Q(x),t\right)\\ \leq \begin{cases}\nu'\left(\lambda,t\,|(r+s)^2-1|\right),\\ \nu'\left(\left[\lambda|x|^a+\lambda|x|^b\right],t\begin{bmatrix}|(r+s)^2-(r+s)^a|\\ +|(r+s)^2-(r+s)^b|\end{bmatrix}\right),\\ \nu'\left(\lambda|x|^{a+b},t\,|(r+s)^2-(r+s)^{a+b}|\right),\\ \nu'\left(2\lambda|x|^{a+b},t\,|(r+s)^2-(r+s)^{a+b}|\right)\end{cases}\end{array}\right\}$$

generalized Ulam - Hyers stability of the functional equation (1).

4. IFNS: Stability Results : Fixed Point Method

In this section, we apply a fixed point method for achieving stability of the quadratic functional equation (1). Here, we present the upcoming result due to Margolis and Diaz [16] for fixed point theory.

Theorem 4.1 [16] Suppose that for a complete generalized metric space (Ω, δ) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then, for each given $x \in \Omega$, either

$$d(T^n x, T^{n+1}x) = \infty, \quad \forall \quad n \geq 0,$$

or there exists a natural number n_0 such that

(FP1) $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;

(FP2) The sequence $(T^n x)$ is convergent to a fixed to a fixed point y^* of T

(FP3) y^* is the unique fixed point of T in the set $\Delta = \{y \in \Omega : d(T^{n_0}x, y) < \infty\}$;

(FP4) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.

Using the above theorem, we now obtain the generalized Ulam - Hyers stability of the functional equation (1).

Theorem 4.2 Let $g : X \to Y$ be a mapping for which there exists a function $K : X \times X \to Z$ with the double condition

$$\left.\begin{array}{l}\lim_{n\to\infty}\mu'\left(K\left(\chi_i^n x,\chi_i^n y\right),\chi^n t\right)=1\\ \lim_{n\to\infty}\nu'\left(K\left(\chi_i^n x,\chi_i^n y\right),\chi^n t\right)=0\end{array}\right\}$$

(4.1)

for all $x, y \in X$ and all $t > 0$ where

$$\chi_i=\begin{cases} r+s & if \quad i=0\\ \dfrac{1}{r+s} & if \quad i=1\end{cases}$$

(4.2)

and satisfying the double functional inequality

$$\left.\begin{array}{l}\mu\left(Dg_{rx}^{sy}(x,y),t\right)\geq\mu'\left(K(x,y),t\right)\\ \nu\left(Dg_{rx}^{sy}(x,y),t\right)\leq\nu'\left(K(x,y),t\right)\end{array}\right\}$$

(4.3)

for all $x, y \in X$ and all $t > 0$. If there exists $L = L(i)$ such that the function

$$\tau(x)=K\left(\frac{x}{r+s},\frac{x}{r+s}\right),$$

(4.4)

has the property

$$\left.\begin{array}{l}\mu'\left(L\dfrac{\tau(\chi_i x)}{\chi_i^2},t\right)=\mu'\left(\tau(x),t\right)\\ \nu'\left(L\dfrac{\tau(\chi_i x)}{\chi_i^2},t\right)=\nu'\left(\tau(x),t\right)\end{array}\right\}$$

(4.5)

for all $x \in X$ and all $t > 0$, then there exists a unique quadratic function $Q : X \to Y$ satisfying the functional equation (1) and

$$\left.\begin{array}{l}\mu\left(g(x)-Q(x),t\right)\geq\mu'\left(\dfrac{L^{1-i}}{1-L}\tau(x),t\right)\\ \nu\left(g(x)-Q(x),t\right)\leq\nu'\left(\dfrac{L^{1-i}}{1-L}\tau(x),t\right)\end{array}\right\}$$

(4.6)

for all $x \in X$ and all $t > 0$.

for all $x \in X$ and all $t > 0$.

Proof. Consider the set
$\Lambda = \{h \mid h : X \to Y, h(0) = 0\}$
and introduce the generalized metric on Λ,

$$d(h,f)=\inf\left\{\begin{array}{l}L\in(0,\infty):\\ \left\{\begin{array}{l}\mu(h(x)-g(x),t)\geq\mu'(L\tau(x),t),x\in X,t>0\\ \nu(h(x)-g(x),t)\leq\nu'(L\tau(x),t),x\in X,t>0\end{array}\right\}\end{array}\right\} \quad (4.7)$$

It is easy to see that (4.7) is complete with respect to the defined metric. Define $J : \Lambda \to \Lambda$ by

$$Jh(x)=\frac{1}{\chi_i^2}h(\chi_i x),$$

for all $x \in X$. Now, from (7) and $h, f \in \Lambda$ and all $t > 0$, we have

$$\begin{cases}\mu(h(x)-g(x),t)\geq\mu'(\tau(x),t);\\ \mu(\dfrac{1}{\chi_i^2}h(\chi_i x)-\dfrac{1}{\chi_i^2}g(\chi_i x),t)\geq\mu'(\tau(\chi_i x),\chi_i^2 t);\\ \mu(Jh(x)-Jg(x),t)\geq\mu'(L\tau(x),t);\end{cases}$$

and

$$\begin{cases}\nu(h(x)-g(x),t)\leq\nu'(\tau(x),t);\\ \nu(\dfrac{1}{\chi_i^2}h(\chi_i x)-\dfrac{1}{\chi_i^2}g(\chi_i x),t)\leq\nu'(\tau(\chi_i x),\chi_i^2 t);\\ \nu(Jh(x)-Jg(x),t)\leq\nu'(L\tau(x),t).\end{cases}$$

This implies J is a strictly contractive mapping on Λ with Lipschitz constant L.

It follows from (4.7),(4.6), we reach and (4.5) for the case $i = 0$, we reach

$$\left\{L^{1-0}\in(0,\infty):\left\{\begin{array}{l}\left\{\begin{array}{l}\mu(g((r+s)x)-(r+s)^2g(x),t)\\ \qquad\geq\mu'\left(K(x,x),t\right)\\ \mu(\dfrac{g((r+s)x)}{(r+s)^2}-g(x),t)\\ \qquad\geq\mu'\left(K(x,x),(r+s)^2t\right)\\ \mu(Jg(x)-g(x),t)\\ \qquad\geq\mu'\left(L\,\tau(x),t\right)\end{array}\right.\\ \left\{\begin{array}{l}\nu(g((r+s)x)-(r+s)^2g(x),t)\\ \qquad\leq\nu'\left(K(x,x),t\right)\\ \nu(\dfrac{g((r+s)x)}{(r+s)^2}-g(x),t)\\ \qquad\leq\nu'\left(K(x,x),(r+s)^2t\right)\\ \nu(Jg(x)-g(x),t)\\ \qquad\leq\nu'\left(L\,\tau(x),t\right)\end{array}\right.\end{array}\right\}\right.$$

(4.8)

for all $x \in X$ and all $t > 0$. Again by interchanging x into $\dfrac{x}{(r+s)}$ in (4.8) and (4.5) for the case $i = 1$, we get

$$\left\{ L^{1-1} \in (0,\infty): \begin{cases} \mu(g(x)-(r+s)^2 g\left(\frac{x}{(r+s)}\right),t) \\ \geq \mu'\left(K\left(\frac{x}{(r+s)},\frac{x}{(r+s)}\right),t\right) \\ \mu(g(x)-Jg(x),t) \geq \mu'\left(\tau(x),t\right) \end{cases} \begin{cases} \nu(g(x)-(r+s)^2 g\left(\frac{x}{(r+s)},\frac{x}{(r+s)}\right),t) \\ \leq \nu'\left(K\left(\frac{x}{(r+s)},\frac{x}{(r+s)}\right),t\right) \\ \nu(g(x)-Jg(x),t) \leq \nu'\left(\tau(x),t\right) \end{cases} \right\} \quad (4.9)$$

for all $x \in X$ and all $t > 0$. Thus, from (4.8) and (4.9), we arrive

$$\inf \left\{ \begin{matrix} L^{1-i} \in (0,\infty): \\ \left\{ \begin{matrix} \mu(g(x)-Jg(x),t) \geq \mu'(L^{1-i}\ \tau(x),t), \\ \nu(g(x)-Jg(x),t) \leq \nu'(L^{1-i}\ \tau(x),t), \end{matrix} \right\} \end{matrix} \right\}$$

(4.10)

Hence property (FP1) holds. By (FP2), it follows that there exists a fixed point Q of J in Λ such that

$$\lim_{n\to\infty} \mu\left(\frac{g(\chi_i^n x)}{\chi_i^{2n}} - Q(x),t\right) = 1,$$

$$\lim_{n\to\infty} \nu\left(\frac{g(\chi_i^n x)}{\chi_i^{2n}} - Q(x),t\right) = 0$$

for all $x \in X$ and all $t > 0$. To order to prove $Q: X \to Y$ is quadratic the proof is similar to that of Theorem .

By (FP3), Q is the unique fixed point of J in the set $\Delta = \{Q \in \Lambda : d(g, A) < \infty\}$, Q is the unique function such that

$$\left. \begin{matrix} \mu(g(x)-Q(x),t) \geq \mu'(L^{1-i}\ \tau(x),t), \\ \nu(g(x)-Q(x),t) \leq \nu'(L^{1-i}\ \tau(x),t), \end{matrix} \right\}$$

for all $x \in X$ and and all $t > 0$. Finally by (FP4), we obtain

$$\left. \begin{matrix} \mu(g(x)-Q(x),t) \geq \mu'\left(\frac{L^{1-i}}{1-L}\tau(x),t\right) \\ \nu(g(x)-Q(x),t) \leq \nu'\left(\frac{L^{1-i}}{1-L}\tau(x),t\right) \end{matrix} \right\}$$

for all $x \in X$ and all $t > 0$. So, the proof is complete.

The next corollary is a direct consequence of Theorem 4.2 which shows that (1.1) can be stable.

Corollary 4.3 Suppose that a function $g: X \to Y$ satisfies the double inequality

$$\left. \begin{matrix} \mu\left(Dg_{rx}^{sy}(x,y),t\right) \\ \geq \begin{cases} \mu'\left(\lambda,t\right), & \\ \mu'\left(\lambda\left(|x|^a+|y|^a\right),t\right), & a \neq 2 \\ \mu'\left(\lambda|x|^a|y|^a,t\right), & 2a \neq 2 \\ \mu'\left(\lambda\left\{|x|^a|y|^a+\left(|x|^{2a}+|y|^{2a}\right)\right\},t\right), & 2a \neq 2 \end{cases} \\ \nu\left(Dg_{rx}^{sy}(x,y),t\right) \\ \leq \begin{cases} \nu'\left(\lambda,t\right), & \\ \nu'\left(\lambda\left(|x|^a+|y|^a\right),t\right), & a \neq 2 \\ \nu'\left(\lambda|x|^a|y|^a,t\right), & 2a \neq 2 \\ \nu'\left(\lambda\left\{|x|^a|y|^a+\left(|x|^{2a}+|y|^{2a}\right)\right\},t\right), & 2a \neq 2 \end{cases} \end{matrix} \right\} \quad (4.11)$$

for all $x, y \in X$ and all $t > 0$, where λ, a are constants with $\lambda > 0$. Then there exists a unique quadratic mapping $Q: X \to Y$ such that the double inequality

$$\mu\left(g(x)-Q(x),t\right) \geq \begin{cases} \mu'\left(\frac{(r+s)^2\lambda}{|(r+s)^2-1|},t\right) \\ \mu'\left(\frac{2\lambda|x|^a}{|(r+s)^2-(r+s)^a|},t\right) \\ \mu'\left(\frac{\lambda|x|^{2a}}{(r+s)^2-(r+s)^{2a}},t\right) \\ \mu'\left(\frac{3\lambda|x|^{2a}}{(r+s)^2-(r+s)^{2a}},t\right) \end{cases}$$

$$\nu\left(g(x)-Q(x),t\right) \leq \begin{cases} \nu'\left(\frac{(r+s)^2\lambda}{|(r+s)^2-1|},t\right) \\ \nu'\left(\frac{2\lambda|x|^a}{|(r+s)^2-(r+s)^a|},t\right) \\ \nu'\left(\frac{\lambda|x|^{2a}}{(r+s)^2-(r+s)^{2a}},t\right) \\ \nu'\left(\frac{3\lambda|x|^{2a}}{(r+s)^2-(r+s)^{2a}},t\right) \end{cases}$$

(4.12)

holds for all $x \in X$ and all $t > 0$.

Proof. Set

$$\mu'\left(K(\chi_i^n x, \chi_i^n y), \chi_i^{2n} t\right)$$
$$= \begin{cases} \mu'\left(\lambda, \chi_i^{2n} t\right), \\ \mu'\left(\lambda\left(|x|^a + |y|^a\right), \chi_i^{2n-a} t\right), \\ \mu'\left(\lambda |x|^a |y|^a, \chi_i^{2n-2a} t\right), \\ \mu'\left(\lambda\left\{|x|^a |y|^a + \left(|x|^{2a} + |y|^{2a}\right)\right\}, \chi_i^{2n-2a} t\right), \end{cases}$$
$$= \begin{cases} \to 1 \; as \; n \to \infty \\ \to 1 \; as \; n \to \infty \\ \to 1 \; as \; n \to \infty \\ \to 1 \; as \; n \to \infty \end{cases}$$

$$\nu'\left(K(\chi_i^n x, \chi_i^n y), \chi_i^{2n} t\right)$$
$$= \begin{cases} \nu'\left(\lambda, \chi_i^{2n} t\right), \\ \nu'\left(\lambda\left(|x|^a + |y|^a\right), \chi_i^{2n-a} t\right), \\ \nu'\left(\lambda |x|^a |y|^a, \chi_i^{2n-2a} t\right), \\ \nu'\left(\lambda\left\{|x|^a |y|^a + \left(|x|^{2a} + |y|^{2a}\right)\right\}, \chi_i^{2n-2a} t\right), \end{cases}$$
$$= \begin{cases} \to 0 \; as \; n \to \infty \\ \to 0 \; as \; n \to \infty \\ \to 0 \; as \; n \to \infty \\ \to 0 \; as \; n \to \infty \end{cases}$$

for all $x \in X$ and all $t > 0$. Thus, the relation (4.1) holds. It follows from (4.4), (4.5) and (4.11)

$$\mu'\left(K\left(\frac{x}{(r+s)}, \frac{x}{(r+s)}\right), t\right) = \begin{cases} \mu'(\lambda, t) \\ \mu'\left(\dfrac{2\lambda |x|^a}{|r+s|^a}, t\right) \\ \mu'\left(\dfrac{\lambda |x|^{2a}}{|r+s|^{2a}}, t\right) \\ \mu'\left(\dfrac{3\lambda |x|^{2a}}{|r+s|^{2a}}, t\right) \end{cases}$$

and

$$\nu'\left(K\left(\frac{x}{(r+s)}, \frac{x}{(r+s)}\right), t\right) = \begin{cases} \nu'(\lambda, t) \\ \nu'\left(\dfrac{2\lambda |x|^a}{|r+s|^a}, t\right) \\ \nu'\left(\dfrac{\lambda |x|^{2a}}{|r+s|^{2a}}, t\right) \\ \nu'\left(\dfrac{3\lambda |x|^{2a}}{|r+s|^{2a}}, t\right) \end{cases}$$

for all $x, y \in X$ and all $t > 0$. Also from (4.5), we have

$$\mu'\left(\frac{\tau(\chi_i x)}{\chi_i^2}, t\right) = \begin{cases} \mu'\left(\chi_i^{-2} \lambda, t\right) \\ \mu'\left(\chi_i^{a-2} \, 2\lambda |x|^a, t\right) \\ \mu'\left(\chi_i^{2a-2} \lambda |x|^{2a}, t\right) \\ \mu'\left(\chi_i^{2a-2} \, 3\lambda |x|^{2a}, t\right) \end{cases}$$

and

$$\nu'\left(\frac{\tau(\chi_i x)}{\chi_i^2}, t\right) = \begin{cases} \nu'\left(\chi_i^{-2} \lambda, t\right) \\ \nu'\left(\chi_i^{a-2} \, 2\lambda |x|^a, t\right) \\ \nu'\left(\chi_i^{2a-2} \lambda |x|^{2a}, t\right) \\ \nu'\left(\chi_i^{2a-2} \, 3\lambda |x|^{2a}, t\right) \end{cases}$$

for all $x \in X$ and all $t > 0$. Hence, the inequality (4.6) is true for

L	$i=0$	L	$i=1$
1. $(r+s)^2$		$(r+s)^{-2}$	
2. $(r+s)^{(a-2)}$	$a<1$	$(r+s)^{(2-a)}$	$a>1$
3. $(r+s)^{(2a-2)}$	$2a<1$	$(r+s)^{(2-2a)}$	$2a>1$
4. $(r+s)^{(2a-2)}$	$2a<1$	$(r+s)^{(2-2a)}$	$2a>1.$

Now, for condition $1.$ and $i=0$, we have

$$\left.\begin{aligned} \mu\left(g(x) - Q(x), t\right) &\geq \mu'\left(\frac{((r+s)^2)^{1-0}}{1-(r+s)^2}\tau(x), t\right) \\ &= \mu'\left(\frac{(r+s)^2 \lambda}{1-(r+s)^2}, t\right) \\ \nu\left(g(x) - Q(x), t\right) &\leq \nu'\left(\frac{((r+s)^2)^{1-0}}{1-(r+s)^2}\tau(x), t\right) \\ &= \nu'\left(\frac{(r+s)^2 \lambda}{1-(r+s)^2}, t\right) \end{aligned}\right\}$$

for all $x \in X$ and all $t > 0$. Also, for condition $1.$ and $i=1$, we get

$$\begin{aligned} \mu\left(g(x) - \mathrm{Q}(x), t\right) &\geq \mu'\left(\frac{(((r+s))^{-2})^{1-1}}{1-((r+s))^{-2}}\tau(x), t\right) \\ &= \mu'\left(\frac{(r+s)^2 \lambda}{(r+s)^2 - 1}, t\right) \\ \nu\left(g(x) - \mathrm{Q}(x), t\right) &\leq \nu'\left(\frac{(((r+s))^{-2})^{1-1}}{1-((r+s))^{-2}}\tau(x), t\right) \\ &= \nu'\left(\frac{(r+s)^2 \lambda}{(r+s)^2 - 1}, t\right) \end{aligned}$$

for all $x \in X$ and all $t > 0$. Again, for condition $2.$ and $i=0$, we obtain

$$\mu\big(g(x)-Q(x),t\big) \geq \mu'\left(\frac{((r+s)^{(a-2)})^{1-0}}{1-((r+s)^{(a-2)})}\tau(x),t\right)$$
$$= \mu'\left(\frac{2\lambda\,|x|^a}{(r+s)^2-(r+s)^a},t\right)$$
$$\nu\big(g(x)-Q(x),t\big) \leq \nu'\left(\frac{((r+s)^{(a-2)})^{1-0}}{1-((r+s)^{(a-2)})}\tau(x),t\right)$$
$$= \nu'\left(\frac{2\lambda\,|x|^a}{(r+s)^2-(r+s)^a},t\right)$$

for all $x \in X$ and all $t > 0$. Also, for condition 2. and $i = 1$, we arrive

$$\mu\big(g(x)-Q(x),t\big) \geq \mu'\left(\frac{((r+s)^{2-a})^{1-1}}{1-((r+s)^{2-a})}\tau(x),t\right)$$
$$= \mu'\left(\frac{2\lambda\,|x|^a}{(r+s)^a-(r+s)^2},t\right)$$
$$\nu\big(g(x)-Q(x),t\big) \leq \nu'\left(\frac{((r+s)^{2-a})^{1-1}}{1-((r+s)^{2-a})}\tau(x),t\right)$$
$$= \nu'\left(\frac{2\lambda\,|x|^a}{(r+s)^a-(r+s)^2},t\right)$$

for all $x \in X$ and all $t > 0$. The rest of the proof is similar to that of previous cases. This finishes the proof.

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