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MOMENT INEQUALITIES FOR SOME BIVARIATE AGEING CLASSES

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A B S T R A C T **RESEARCH ARTICLE**

In this paper, we derive the Moments Inequalities for some bivariate ageing classes

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1. Introduction

In this paper we establish inequalities for the moments of bivariate ageing classes, namely, Bivariate Increasing Failure Rate (BIFR), Bivariate New Better than Used in Expection (BNBUE), Bivariate Harmonic New Better than Used in Expectation (BHNBUE), Bivariate New Better than Used in Failure Rate (BNBUFR), Bivariate New Better than Renewal Used (BNBRU), Bivariate New Renewal better than Used (BNRBU), Bivariate Renewal New is Better than Renewal Used in Expectation (BRNBRUE), Bivariate New Better than Used in Laplace transform order (BNBUL), Bivariate Exponential Better than Used (BEBU), Bivariate Exponential Better than Used in Convex order (BEBUC(2)), Bivariate Exponential Better than Used in Convex Average (BEBUCA), Certain class of life distributions and their variations have been introduced. The application of these bivariate classes of life distributions can be seen in engineering, social and biological sciences. Reliability analysis have shown a growing interest in modeling, survival data using classification of life distributions based on some aspects of aging.

2. Preliminaries

In this section, we give below the definition of various bivariate stochastic ageing classes that are required for further discussion.

Definition 2.1 A bivariate random variable (X,Y) or its distribution $\overline{F}(x,y)$ is said to have Bivariate Increasing Failure Rate (BIFR) if

$$\frac{\overline{F}(x+t,y+s)}{\overline{F}(t,s)}$$

is decreasing in t, whenever $x,y \geq 0$.

Definition 2.2 A bivariate random variable (X,Y) or its distribution $\overline{F}(x,y)$ is said to have Bivariate New Better than Used in Expectation (BNBUE) if

$$\int_0^\infty\int_0^\infty \overline{F}(x+t,y+s)dtds \leq \overline{F}(x,y)\int_0^\infty\int_0^\infty \overline{F}(t,s)dtds \;\; for\; x,y \geq 0.$$

Definition 2.3 A bivariate random variable (X,Y) or its distribution $\overline{F}(x,y)$ is said to have Bivariate Harmonic New Better than Used in Expectation (BHNBUE) if

$$\int_t^\infty\int_s^\infty \overline{F}(x,y)dydx \leq \text{«}\; exp\left[-\frac{t+s}{\text{«}}\right],\; for\; t,s \geq 0$$

and « denotes the finite mean given by

$$\text{«} = \int_0^\infty\int_0^\infty \overline{F}(x,y)dydx$$

Definition 2.4 A bivariate random variable (X,Y) or its distribution $\overline{F}(x,y)$ having failure rate $r(x,y)$ is said to have Bivariate New Better than Used in Failure Rate (BNBUFR) if

$$r(0,0) \leq r(x,y) \;\; for\; x,y \geq 0.$$

Definition 2.5 A bivariate random variable (X,Y) or its distribution $\overline{F}(x,y)$ is said to have Bivariate New Better than Renewal Used (BNBRU) if

$$\int_{y+s}^\infty\int_{x+t}^\infty \overline{F}(u,v)dvdu \leq \overline{F}(x,y)\int_t^\infty\int_s^\infty \overline{F}(u,v)dvdu \;\; for\; x,y,t,s \geq 0.$$

Put

$$\overline{W}(x,y) = \frac{1}{\text{«}}\int_y^\infty\int_x^\infty \overline{F}(u,v)dvdu$$

then the above inequality becomes

$$\overline{W}(x+t,y+s) \leq \overline{F}(x,y)\overline{W}(t,s),\; for\; all\; x,y,t,s \geq 0$$

Definition 2.6 A bivariate random variable (X,Y) or its distribution $\overline{F}(x, y)$ is said to have Bivariate New Renewal Better than Renewal Used (BNRBU) if

$$\overline{F}(x+t, y+s) \leq \overline{F}(x, y)\overline{W}(t, s),$$

where

$$\overline{W}(t, s) = \int_s^\infty \int_t^\infty \overline{F}(u, v)dvdu.$$

Definition 2.7 A bivariate random variable (X, Y) or its distribution $\overline{F}(x, y)$ is said to have Bivariate Renewal New Better than Used (BRNBU) if

$$\overline{F}_{t,s}(x, y) \leq \overline{W}_F(x, y)$$

where,

$$\overline{F}_{t,s}(x, y) = \frac{\overline{F}(t+x, s+y)}{\overline{F}(t, s)} : \overline{F} > 0$$

$$\overline{W}_F(x, y) = \frac{1}{«}\int_x^\infty \int_y^\infty \overline{F}(u, v)dvdu \ \ for\ x, y \geq 0$$

Here $«$ denotes the mean of the bivariate life distribution F and is assumed to be finite.

Definition 2.8 A bivariate random variable (x, y) or its distribution $\overline{F}(x, y)$ is said to have Bivariate Renewal New is Better than Renewal Used (BRNBRU) if

$$\overline{W}_F(x+t, y+s) \leq \overline{W}_F(x, y)\overline{W}_F(t, s).$$

For all $x, y, t, s \geq 0$ That is

$$« \int_{x+t}^\infty \int_{y+s}^\infty \overline{F}(u, v)dvdu \leq \left(\int_x^\infty \int_y^\infty \overline{F}(u, v)dvdu\right). \left(\int_t^\infty \int_s^\infty \overline{F}(u, v)dvdu\right),$$

for all $x, y, t, s \geq 0$.

Definition 2.9 A bivariate random variable (X, Y) or its distribution $\overline{F}(x, y)$ is said to have Bivariate New Better than Used in Laplace transform order (BNBUL) if

$$\int_0^\infty \int_0^\infty exp[-ª(x+y)]\overline{F}(x+t, y+s)dydx \leq \overline{F}(t, s)\int_0^\infty \int_0^\infty exp[-ª(x+y)]\overline{F}(x, y)dydx,$$

for all $x, y > 0$ and $ª \geq 0$.

Definition 2.10 A bivariate random variable (X, Y) or its distribution $\overline{F}(x, y)$ is said to haveBivariate Renewal New is Better than Renewal Used in Expectation (BRNBRUE) if

$$2« \int_x^\infty \int_y^\infty \int_u^\infty \int_v^\infty \overline{F}(t, s)dsdt \leq «_{(2)}\int_x^\infty \int_y^\infty \overline{F}(t, s)dsdt.$$

Definition 2.11 A bivariate random variable (X, Y) or its distribution $\overline{F}(x, y)$ is said to have Bivariate Exponential Better than (BEBU) if

$$\overline{F}(x+t, y+s) \leq \overline{F}(t, s).e^{\frac{(x+y)}{«}} \ \ for\ all\ x, t, y, s \geq 0.$$

Definition 2.12 A bivariate random variable (X, Y) or its distribution $\overline{F}(x, y)$ is said to have Bivariate Exponential Better than Used in Convex order Two (BEBUC(2)) if

$$\int_u^\infty \int_v^\infty \overline{F}(x+t, y+s)dsdt \leq « \ exp\left[-\left(\frac{x+y}{«}\right)\right]\int_u^\infty \int_v^\infty \overline{F}(t, s)dsdt,$$

for all $x, y, u, v \geq 0$.

Definition 2.13 A bivariate random variable (X, Y) or its distribution $\overline{F}(x, y)$ is said to have Bivariate Exponential Better than Used Convex order Average (BEBUCA) if

$$\int_0^\infty \int_0^\infty \int_{y+s}^\infty \int_{x+t}^\infty \overline{F}(u, v)dvdudxdy \leq «^2\ \overline{F}(t, s)$$

Put

$$\overline{W}(x+t, y+s) = \int_{y+s}^\infty \int_{x+t}^\infty \overline{F}(u, v)dvdu$$

then the above inequality becomes

$$\int_0^\infty \int_0^\infty \overline{W}(x+t, y+s)dxdy \leq «^2\ \overline{F}(t, s), \ \ for\ all\ t, s \geq 0.$$

Definition 2.14 A bivariate random variable (X, Y) or its distribution $\overline{F}(x, y)$ is said to have Bivariate Overall Decreasing Life class (BODL) if

$$\int_s^\infty \int_t^\infty \overline{W}(x, y)dxdy \leq «\ \overline{W}(t, s),$$

where

$$\overline{W}(t, s) = \frac{1}{«}\int_s^\infty \int_t^\infty \overline{F}(u, v)dvdu$$

and

$$« = \int_0^\infty \int_0^\infty \overline{F}(u, v)dvdu$$

is assumed to be finite.

Definition 2.15 A bivariate random variable (X, Y) or its distribution $\overline{F}(x, y)$ is said to have Bivariate r^{th} Moment

$$«_n = E(X^n Y^n) = n^2\int_0^\infty \int_0^\infty (xy)^{n-1}\overline{F}(x, y)dydx.$$

3. Moment Inequality

In this section we established some theorems on Moment Inequality. We now present a theorem which is useful for further discussion

Theorem 3.1 If (i) F is BIFR with mean $«$ and

$$\overline{G}(x, y) = e^{-\frac{\sqrt{x^2+y^2}}{«}}$$

(ii) $¤(x, y)$ is increasing in x and increasing in y then,

$$\int_0^\infty \int_0^\infty ¤(x, y)\overline{F}(x, y)dxdy \leq \int_0^\infty \int_0^\infty ¤(x, y)\overline{G}(x, y)dxdy$$

Proof. Suppose $¤$ is increasing and F is not identically equal to G.

Since F is BIFR and G is bivariate exponential distribution with common mean $«$.

$\overline{F}$ crosses $\overline{G}$ exactly once from the above,Say at (t_0, s_0). That is ,

$$\overline{F}(t_0,s_0)=\overline{G}(t_0,s_0)\,.$$

Now,

$$\int_0^\infty\int_0^\infty ¤(x,y)\overline{F}(x,y)dxdy-\int_0^\infty\int_0^\infty ¤(x,y)\overline{G}(x,y)dxdy$$

$$\le\int_0^\infty\int_0^\infty[¤(x,y)-¤(t_0,s_0)][\overline{F}(x,y)-\overline{G}(x,y)]dxdy$$

$$\le 0$$

This proves the theorem.

Theorem 3.2 If F is BIFR with r-th moment $«_{(r)}$, then

$$«_{(r)}\begin{cases}\le «^r r! & : r\ge 1,\\ \ge «^r r! & : 0\le r\le 1\end{cases}$$

Proof. Put $¤(x,y)=(xy)^{r-1}$ in the previous theorem then

$$\int_0^\infty\int_0^\infty (xy)^{r-1}\overline{F}(x,y)dxdy$$

$$\le\int_0^\infty\int_0^\infty (xy)^{r-1}\overline{G}(x,y)dxdy\ \ for\ r\ge 1$$

$$(i.e)\ \frac{«_{(r)}}{r}\le\int_0^\infty\int_0^\infty (xy)^{r-1}e^{-\frac{(x+y)}{«}}dxdy$$

$$\le\left[\int_0^\infty (x)^{r-1}e^{-\frac{(x)}{«}}dx\right]\left[\int_0^\infty (y)^{r-1}e^{-\frac{(y)}{«}}dy\right]$$

$$\le «^r(r-1)!$$

$$«(r)\le r!\,«^r$$

This completes the proof of the theorem.

Theorem 3.3 If F is BNBUE then for all integers $r\ge 0$

$$\frac{«(r+2)}{(r+2)}\le\frac{«.«(r+1)}{(r+1)}$$

Proof. Since F is BNBUE, we have

$\overline{W}(x,y)=\int_x^\infty\int_y^\infty\overline{F}(u,v)dvdu$. Thus for all integers $r>0$

$$\int_0^\infty\int_0^\infty x^r y^r\overline{W}(x,y)dydx\le «\int_0^\infty\int_0^\infty x^r y^r\overline{F}(x,y)dydx$$

$$\le «\frac{«_{(r+1)}}{(r+1)^2}$$

$$\le «\frac{«_{(r+1)}}{(r+1)} \tag{3.1}$$

Further

$$\int_0^\infty\int_0^\infty x^r y^r\overline{W}\ x,y\ dydx$$

$$=E\left[\begin{array}{l}\int_0^\infty\int_0^\infty x^r y^r(X-x)(Y-y).\\ I(X>x)I(Y>y)dydx\end{array}\right]$$

$$=E\left[\begin{array}{l}\left(\int_0^\infty x^r(X-x)I(X>x)dx\right)\\ \left(\int_0^\infty y^r(Y-y)I(Y>y)dy\right)\end{array}\right]$$

$$=E\left[\left(X\int_0^X x^r dx\right)\left(Y\int_0^y y^r dy-\int_0^y y^{r+1}dy\right)\right]$$

$$=E\left[\left(X\frac{x^{r+1}}{r+1}\Big|_0^X-\frac{x^{r+2}}{r+2}\Big|_0^X\right)\cdot\left(Y\frac{y^{r+1}}{r+1}\Big|_0^Y-\frac{y^{r+2}}{r+2}\Big|_0^Y\right)\right]$$

$$=E\left[X^{r+2}\left(\frac{1}{r+1}-\frac{1}{r+2}\right)\times Y^{r+2}\left(\frac{1}{r+1}-\frac{1}{r+2}\right)\right]$$

$$=E\left[X^{r+2}Y^{r+2}\right]\left[\frac{r+2-r-1}{(r+1)(r+2)}\right]$$

$$=\frac{«_{(r+2)}}{(r+1)(r+2)}$$

Therefore the inequality (3.1) becomes

$$\frac{«_{(r+2)}}{(r+1)(r+2)}\le\frac{«.«_{(r+1)}}{(r+1)^2}$$

$$\frac{«_{(r+2)}}{(r+2)}\le\frac{«.«_{(r+1)}}{(r+1)}$$

This completes the proof of the theorem.

Theorem 3.4 If F is BHNBUE then for all integers $r\ge 0$ $«_2\le 2«^2$

Proof. Since F is BHNBUE we have

$$\overline{W}(x,y)\le «\exp(-\frac{x+y}{«})$$

$$\int_0^\infty\int_0^\infty x^r y^r\overline{W}(x,y)dydx\le «\int_0^\infty\int_0^\infty x^r y^r\exp(-\frac{x+y}{«})dydx$$

$$=«^{2r+3}(r!)^2$$

But

$$\int_0^\infty\int_0^x x^r y^r\overline{W}(x,y)dydx=\frac{«_{(r+2)}}{(r+1)(r+2)}$$

Therefore

$$\frac{«_{(r+2)}}{(r+1)(r+2)}\le «^{r+2}(r!)$$

$$\frac{«_{(r+2)}}{(r+2)!}\le «^{r+2}$$

If $r=0$

$$\frac{«_2}{2!}=«^2$$

$$«_2=2«^2$$

This completes the proof of the theorem.

Theorem 3.5 Let F be BNBUFR such that for some integers $r,s\ge 0$

$$\frac{«_{r+s+2}}{(r+s+2)}\le\frac{«_{(r+1)}}{(r+1)}\frac{s!}{[r(0,0)]^{s+1}}$$

Proof. Since F is BNBUFR, we have

$$\overline{F}(x+u,y+v)\le\overline{F}(x,y)e^{-r(0,0)\sqrt{u^2+v^2}}.$$

Then for all integers $r,s\ge 0,$

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty x^r y^r u^s v^s\overline{F}(x+u,y+v)dvdudydx$$

$$\leq \left(\int_0^\infty\int_0^\infty x^r y^r \overline{F}(x,y)dydx\right)$$

$$\left(\int_0^\infty\int_0^\infty u^s v^s e^{-r}(0,0)^{\sqrt{u^2+v^2}}\, dvdu\right)$$

$$\leq E\left[\int_0^X\int_0^Y x^r y^r dydx\right]\frac{1}{r(0,0)^{s+1}}(s!)$$

$$\leq \frac{«_{(r+1)}}{r+1}\frac{s!}{[r(0,0)^{s+1}]}$$

(3.2)

Also

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty x^r y^r u^s v^s \overline{F}(x+u,y+v)dvdudydx$$

$$=\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty (xy)^r (uv)^s \overline{F}(x+u,y+v)dvdudydx$$

$$=\int_0^\infty\int_0^\infty\int_0^{~}\int_0^{¥}(~-\ )^r\ \ ^s(¥-¢)^r ¢^s\overline{F}(~,¥)d¥d~d¢d$$

$$=\int_0^\infty\int_0^\infty ~^{r+s+1}¥^{r+s+1}\overline{F}(~,¥)d¥d~$$

$$\int_0^1\int_0^1 (pq)^s(1-pq)^r dqdp$$

$$=\ \ (r+1,s+1)\int_0^\infty\int_0^\infty(~¥)^{r+s+1}\overline{F}(~,¥)d¥d~$$

$$=\frac{«_{(r+s+2)}}{(r+s+2)}\frac{r!s!}{(r+s+1)}$$

Therefore the inequality (3.3) becomes

$$\frac{«_{(r+s+2)}}{(r+s+2)}\leq\frac{«_{(r+1)}}{(r+1)}\frac{(s!)}{[r(0,0)^{s+1}]}$$

This completes the proof of the theorem.

Theorem 3.6 Suppose that F is BNBRU life distribution and it's $(r+3)^{rd}$ moment is finite for some integer $r\geq 0$, then

$$\frac{«_{(m+3)}}{(m+2)(m+3)}$$

$$\leq\sum_{i=0}^m\binom{m}{i}\frac{«_{(i+1)}«_{(m-i+2)}}{(i+1)(m-i+1)(m-i+2)}$$

Proof. Since F is a BNBRU, we have

$$\overline{W}(x,y)=\frac{1}{«}\int_y^\infty\int_x^\infty\overline{F}(u,v)dvdu$$

Then the above inequality becomes

$$\overline{W}(x+t,y+s)\leq\overline{F}(x,y)\overline{W}(t,s)\forall\ x,y,t,s>0$$

Multiplying both side by $(x+t)^m(y+s)^m$ and integrating we get

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty(x+t)^m(y+s)^m\overline{W}(x+t,y+s)dxdydsdt$$

$$\leq\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty(x+t)^m(y+s)^m\overline{F}(x,y)\overline{W}(t,s)dxdydsdt \qquad (3.3)$$

Also

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty(x+t)^m(y+s)^m\overline{W}(x+t,y+s)dxdydsdt$$

$$=\int_0^\infty\int_0^\infty\int_s^\infty\int_t^\infty(z_1z_2)^m\overline{W}(z_1,z_2)dz_1dz_2dsdt$$

$$=\int_0^\infty\int_0^{~}(~\ \ )^m\overline{W}(~,\ \ )\int_0^\infty\int_0 dz_1dz_2d~d$$

$$=\int_0^\infty\int_0^\infty(~\ \ )^{m+1}\overline{W}(~,\ \ )d~d$$

$$=\frac{1}{«}\int_0^\infty\int_0^\infty(~\ \ )^{m+1}\int^\infty\int_{~}^\infty\overline{F}(u,v)dvdud~d$$

$$=\frac{1}{«}\int_0^\infty\int_0^\infty\overline{F}(u,v)\int_0^u\int_0^v(~\ \ )^{m+1}d~d\ \ dvdu$$

$$=\frac{1}{«}\int_0^\infty\int_0^\infty\frac{(uv)^{m+2}}{m+2}\overline{F}(u,v)dvdu$$

$$=\frac{1}{«(m+2)}\int_0^\infty\int_0^\infty(uv)^{m+2}\overline{F}(u,v)dvdu$$

$$=\frac{«_{(m+3)}}{«(m+2)(m+3)}$$

(3.4)

Also $\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty(x+t)^m(y+s)^m\overline{F}(x,y)\overline{W}(t,s)dxdydsdt$

$$=\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty(x+t)^m(y+s)^m\overline{F}(x,y)\overline{W}(t,s)dxdydsdt$$

$$=\sum_{i=0}^m\binom{m}{i}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty x^it^{m-i}y^is^{m-i}\overline{F}(x,y)\overline{W}(t,s)dxdydsdt$$

$$=\sum_{i=0}^m\binom{m}{i}\left[\begin{array}{l}\left(\int_0^\infty\int_0^\infty(xy)^i\overline{F}(x,y)dxdy\right)\\ \left(\int_0^\infty\int_0^\infty(ts)^{m-i}\overline{W}(t,s)dsdt\right)\end{array}\right]$$

$$=\sum_{i=0}^m\binom{m}{i}\frac{«_{(i+1)}}{(i+1)}\left[\frac{1}{«}\int_0^\infty\int_0^\infty(ts)^{m-i}\int_s^\infty\int_t^\infty\overline{F}(u,v)dvdudsdt\right]$$

$$=\sum_{i=0}^m\binom{m}{i}\frac{«_{(i+1)}}{(i+1)}\frac{1}{«}\int_0^\infty\int_0^\infty\overline{F}(t,s)\int_0^s\int_0^t(uv)^{m-i}dvdudsdt$$

$$=\sum_{i=0}^m\binom{m}{i}\frac{«_{(i+1)}}{(i+1)}\frac{1}{«}\int_0^\infty\int_0^\infty\frac{(ts)^{m-i+1}}{(m-i+1)}\overline{F}(t,s)dsdt$$

$$=\sum_{i=0}^m\binom{m}{i}\frac{«_{(i+1)}}{(i+1)}\frac{1}{«}\frac{«_{(m-i+2)}}{(m-i+1)(m-i+2)} \qquad (3.5)$$

Using (3.4) and (3.5) in (3.3) we get

$$\frac{«_{(m+3)}}{(m+2)(m+3)}=\sum_{i=0}^m\binom{m}{i}\frac{«_{(i+1)}«_{(m-i+2)}}{(i+1)(m-i+1)(m-i+2)}$$

This completes the proof of the theorem.

Theorem 3.7 For all non-negative integer $r\geq 0$ and F is BNRBU we get

$$\frac{«_{(m+2)}}{(m+2)}\leq\sum_{i=0}^m\binom{m}{i}\frac{«_{(m-i+1)}}{(m-i+1)}\frac{«_{(i+2)}}{(i+1)(i+2)}$$

Proof. Since F is a BNRBU, we have

$$\overline{W}(t,s)=\int_s^\infty\int_t^\infty\overline{F}(u,v)dvdu$$

Multiplying both sides by $[(x+t)(y+s)]^m$ and integrating, we have

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty (x+t)^m (y+s)^m \overline{F}(x+t,y+s)dxdydtds$$

$$\le \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty (x+t)^m (y+s)^m \overline{F}(x,y)\overline{W}(t,s)dxdydtds \qquad (3.6)$$

Also

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty (x+t)^m (y+s)^m \overline{F}(x+t,y+s)dxdydtds$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty (x+t)^m (y+s)^m \overline{F}(x+t,y+s)dxdydtds$$

$$= \int_0^\infty\int_0^\infty\int_0^X\int_0^Y (uv)^m \overline{F}(t,s)dvdudtds$$

$$= E\left[\int_0^X\int_0^Y (xy)^{m+1} I[X-x,Y-y]dxdy\right]$$

$$= \frac{E(XY)^{m+2}}{m+2}$$

$$= \frac{«_{(m+2)}}{(m+2)} \qquad (3.7)$$

Also

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty (x+t)^m (y+s)^m \overline{F}(x,y)\overline{W}(t,s)dxdydtds$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty (x+t)^m (y+s)^m \overline{F}(x,y)\overline{W}(t,s)dxdydtds$$

$$= \sum_{i=0}^{m}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \binom{m}{i} x^{m-i}t^i y^{m-i}s^i \overline{F}(x,y).$$

$$\left(\int_s^\infty\int_t^\infty \overline{F}(u,v)dvdu\right)dxdydsdt$$

$$= \sum_{i=0}^{m}\binom{m}{i}\int_0^\infty\int_0^\infty (xy)^{m-1}\overline{F}(x,y)dxdy.$$

$$\int_0^\infty\int_0^\infty (ts)^i \int_s^\infty\int_t^\infty \overline{F}(u,v)dvdudsdt$$

$$= \sum_{i=0}^{m}\binom{m}{i}\frac{«_{(m-i+1)}}{(m-i+1)}\int_0^\infty\int_0^\infty (ts)^i \int_s^\infty\int_t^\infty \overline{F}(u,v)dvdudsdt$$

$$= \sum_{i=0}^{m}\binom{m}{i}\frac{«_{(m-i+1)}}{(m-i+1)}\int_0^\infty\int_0^\infty \overline{F}(x,y)\int_0^X\int_0^Y (u,v)^i dvdudxdy$$

$$= \sum_{i=0}^{m}\binom{m}{i}\frac{«_{(m-i+1)}}{(m-i+1)}\int_0^\infty\int_0^\infty \frac{(XY)^{i+1}}{(i+1)}\overline{F}(x,y)dxdy$$

$$= \sum_{i=0}^{m}\binom{m}{i}\frac{«_{(m-i+1)}}{(m-i+1)}\frac{1}{(i+1)}\frac{«_{(i+2)}}{(i+2)}$$

$$= \sum_{i=0}^{m}\binom{m}{i}\frac{«_{(m-i+1)}}{(m-i+1)}\frac{«_{(i+2)}}{(i+1)(i+2)} \qquad (3.8)$$

Using (3.7) and (3.8) the inequality (3.6) becomes

$$\frac{«_{(m+2)}}{(m+2)} \le \sum_{i=0}^{m}\binom{m}{i}\frac{«_{(m-i+1)}}{(m-i+1)}\frac{«_{(i+2)}}{(i+1)(i+2)}$$

This completes the proof of the theorem.

Theorem 3.8 For all non-negative integer $r \ge 0$ and F is BRNBU we get

$$\frac{«_{(m+3)}}{(m+2)(m+3)} \le \frac{1}{«}\sum_{i=0}^{m}\binom{m}{i}\frac{1}{(m-i+1)(i+1)}\frac{«_{(m-i+2)}«_{(i+2)}}{(m-i+2)(i+2)}$$

Proof. Since F is a BRNBU, we have

$$\overline{W}(x+t,y+s) \le \overline{W}(x,y)\overline{W}t,s)$$

Multiplying both sides by $(x+t)^m(y+s)^m$ and integrating over $(0,\infty)$, we have

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty (x+t)^m (y+s)^m \overline{W}(x+t,y+s)dxdydsdt$$

$$\le \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty (x+t)^m (y+s)^m \overline{W}(x,y)\overline{W}(t,s)dxdydsdt$$

Also

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty (x+t)^m (y+s)^m \overline{W}(x+t,y+s)dxdydsdt$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty (x+t)^m (y+s)^m \overline{W}(x+t,y+s)dxdydsdt$$

$$= \frac{1}{«}\int_0^\infty\int_0^\infty \overline{F}(x,y)\int_0^X\int_0^Y (uv)^{m+1}dvdudxdy$$

$$= \frac{1}{«}E\left[\int_0^\infty\int_0^\infty \frac{(XY)^{m+2}}{m+2} I[X>x,Y>y]dxdy\right]$$

$$= \frac{1}{«}\frac{1}{m+2}E(xy)^{m+2}$$

$$= \frac{«_{(m+3)}}{«(m+2)(m+3)} \qquad (3.10)$$

where I is a indicator function.

Also

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty (x+t)^m (y+s)^m \overline{W}(x,y)\overline{W}(t,s)dxdydsdt$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty (x+t)^m (y+s)^m \overline{W}(x,y)\overline{W}(t,s)dxdydsdt$$

$$= \frac{1}{«^2}\sum_{i=0}^{m}\binom{m}{i}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty x^{m-i}t^i y^{m-i}s^i$$

$$\overline{W}(x,y)\overline{W}(t,s)dxdydsdt$$

$$= \frac{1}{«^2}\sum_{i=0}^{m}\binom{m}{i}\left(\int_0^\infty\int_0^\infty (xy)^{m-i}\overline{W}(x,y)dxdy\right)$$

$$\left(\int_0^\infty\int_0^\infty (ts)^i \overline{W}(t,s)dsdt\right)$$

$$= \frac{1}{«^2}\sum_{i=0}^{m}\binom{m}{i}\int_0^\infty\int_0^\infty \frac{(xy)^{m-i+1}}{(m-i+1)} I(X>x,Y>y)dxdy$$

$$E\left[\int_0^\infty\int_0^\infty \frac{(ts)^{i+1}}{i+1} I(T>t,S>s)dsdt\right]$$

$$= \frac{1}{«^2}\sum_{i=0}^{m}\binom{m}{i}\frac{1}{(m-i+1)(i+1)}\frac{«_{(m-i+2)}}{(m-i+2)}\frac{«_{(i+2)}}{(i+2)} \qquad (3.11)$$

Using (3.10) and (3.11) the inequality (3.9) become

$$\frac{«_{(m+3)}}{(m+2)(m+3)} =$$

$$\frac{1}{«}\sum_{i=0}^{m}\binom{m}{i}\frac{1}{(m-i+1)(i+1)}\frac{«_{(m-i+2)}«_{(i+2)}}{(m-i+2)(i+2)}$$

This completes the proof of the theorem.

Theorem 3.9 If F is BRNBRU then

$$\frac{«.«_{(m+n+3)}}{(m+n+3)!} \le \frac{«_{(m+2)}«_{(n+2)}}{(m+2)!(n+2)!}$$

Proof. Let

$$\overline{V}(x,y) = \int_x^\infty \int_y^\infty \overline{F}(u,v)dvdu.$$

Since F is a BRNBRU, we have

$$«\overline{V}(x+t, y+s) \le \overline{V}(x,y)\overline{V}(t,s)$$

Multiplying by $(xy)^m(ts)^n; m,n>0$ (integers) and integrating over $(0,\infty)$ with respect to x,y,t,s we get,

$$«\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty (xy)^m(ts)^m\overline{V}(x+t,y+s)dxdydsdt$$
$$\le \left(\int_0^\infty\int_0^\infty (xy)^m\overline{V}(x,y)dxdy\right)\left(\int_0^\infty\int_0^\infty (ts)^n\overline{V}(t,s)dsdt\right) \quad (3.12)$$

Also

$$«\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty (xy)^m(ts)^m\overline{V}(x+t,y+s)dxdydsdt$$
$$= «\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty (xy)^m(ts)^m\overline{V}(x+t,y+s)dxdydsdt$$
$$= \frac{«}{(m+1)(m+2)}E\left[\begin{array}{l}\int_0^\infty\int_0^\infty (ts)^n E(XY-ts)^{m+2}\\ I(X>t, Y>s)dsdt\end{array}\right]$$
$$= \frac{«}{(m+1)(m+2)}E\left[\int_0^X\int_0^Y (ts)^n(XY-ts)^{m+2}dsdt\right]$$
$$= \frac{«}{(m+1)(m+2)}E\left[(XY)^{m+n+3}\int_0^1\int_0^1 (\sim\ \ )^n(1-\sim\ \ )^{m+2}d\sim d\ \ \right]$$
$$= \frac{«}{(m+1)(m+2)}E\left[(XY)^{m+n+3}\right]\frac{\Gamma(x+1)\Gamma(m+3)}{\Gamma(m+n+4)}$$
$$= \frac{«.«_{(m+n+3)}}{(m+1)(m+2)}\frac{n!(m+2)!}{(m+n+3)!}$$
$$= \frac{m!n!«.«_{(m+n+3)}}{(m+n+3)!} \quad (3.13)$$

Consider

$$\int_0^\infty\int_0^\infty (xy)^m\overline{V}(x,y)dydx =$$
$$E\left[\int_0^\infty\int_0^\infty (xy)^m(XY-xy)I(X>x, Y>ydydx)\right]$$
$$= E\left[XY\int_0^X\int_0^Y (xy)^m dydx - \int_0^X\int_0^Y (xy)^{m+1}dydx\right]$$
$$= E\left[\frac{(XY)^{m+2}}{(m+1)} - \frac{(XY)^{m+2}}{(m+2)}\right]$$
$$= E\left[(XY)^{m+2}\left(\frac{1}{(m+1)} - \frac{1}{(m+2)}\right)\right]$$
$$= \frac{1}{(m+1)(m+2)}E[(XY)^{m+2}]$$

$$\int_0^\infty\int_0^\infty (xy)^m\overline{V}(x,y)dydx = \frac{«_{(m+2)}}{(m+1)(m+2)} \quad (3.14)$$

Similarly

$$\int_0^\infty\int_0^\infty (ts)^n\overline{V}(t,s)dsdt = \frac{«(n+2)}{(n+1)(n+2)} \quad (3.15)$$

Using (3.13),(3.14) and (3.15) in (3.12) we get

$$\frac{m!n!«.«_{(m+n+3)}}{(m+n+3)!} \le \frac{«_{(m+2)}}{(m+1)(m+2)}\frac{«_{(n+2)}}{(n+1)(n+2)}$$
$$\frac{«.«_{(m+n+3)}}{(m+n+3)!} \le \frac{«_{(m+2)}«_{(n+2)}}{(m+2)!(n+2)!}$$

This completes the proof of the theorem.

Theorem 3.10 If F is BRNBRUE then

$$2«\frac{«_{(m+2)}}{(m+1)!} \le \frac{«_{(2)}«_{(m+3)}}{(m+3)}$$

Proof. If F is a BRNBRUE, we have

$$2«\int_y^\infty\int_x^\infty \overline{W}(u,v)dvdu \le «_{(2)}\overline{W}(x,y)$$

Multiplying both sides by $(xy)^r$ and inequality

$$2«\int_0^\infty\int_0^\infty\int_y^\infty\int_x^\infty (xy)^m\overline{W}(u,v)dvdudxdy$$
$$\le «_{(2)}\int_0^\infty\int_0^\infty (xy)^m\overline{W}(x,y)dxdy \quad (3.16)$$

Consider

$$\int_0^\infty\int_0^\infty (xy)^m\overline{W}(x,y)dxdy =$$
$$\int_0^\infty\int_0^\infty (xy)^m\left(\int_y^\infty\int_x^\infty \overline{F}(u,v)dudv\right)dxdy$$
$$= \int_0^\infty\int_0^\infty \overline{F}(x,y)\int_0^y\int_0^x (uv)^m dudvdxdy$$
$$= \int_0^\infty\int_0^\infty \frac{(xy)^{m+1}}{(m+1)^2}\overline{F}(x,y)dxdy$$
$$= \frac{1}{(m+1)^2}\int_0^\infty\int_0^\infty (xy)^{m+1}\overline{F}(x,y)dxdy$$

$$\int_0^\infty\int_0^\infty (xy)^m\overline{W}(x,y)dxdy = \frac{1}{(m+1)^2}\frac{«_{(m+2)}}{(m+2)} \quad (3.17)$$

and

$$\int_0^\infty\int_0^\infty\int_y^\infty\int_x^\infty (xy)^m\overline{W}(u,v)dvdudxdy$$
$$= \int_0^\infty\int_0^\infty \overline{W}(x,y)\frac{(xy)^{m+1}}{m+1}dxdy$$
$$= \frac{1}{(m+1)}E\left[\begin{array}{l}\int_0^\infty\int_0^\infty (xy)^{m+1}(XY-xy).\\ \qquad I(X>x, Y>y)dxdy\end{array}\right]$$
$$= \frac{1}{(m+1)}E\left[\begin{array}{l}XY\int_0^Y\int_0^X (xy)^{m+1}dydx\\ \qquad -\int_0^Y\int_0^X (xy)^{m+2}dydx\end{array}\right]$$

$$= \frac{1}{(m+1)} E\left[\frac{(XY)^{m+2}}{m+2} - \frac{(XY)^{m+3}}{m+3}\right]$$

$$= \frac{1}{(m+1)} «_{(m+3)}\left(\frac{1}{(m+2)} - \frac{1}{(m+3})\right)$$

$$= \frac{«_{(m+3)}}{(m+1)(m+2)(m+3)} \quad (3.18)$$

Using (3.17) and (3.18) the inequality (3.16) becomes

$$2« \frac{«_{(m+2)}}{(m+1)^2(m+2)} \leq \frac{«_{(2)} «_{(m+3)}}{(m+1)(m+2)(m+3)}$$

$$2« \frac{«_{(m+2)}}{(m+1)} \leq \frac{«_{(2)} «_{(m+3)}}{(m+3)}$$

This completes the proof of the theorem.

Theorem 3.11 Let $¤(ª) = \int_0^\infty\int_0^\infty e^{-ª(x+y)} dF(x+y)$ if F is BNBUL. Then for all integer $s, r \geq 0$

$$\frac{(-1)^{m+1} m!}{ª^{m+1}}\left[1 - ¨(ª)\right] + \frac{m!}{ª^{m+1}} \sum_{i=0}^{m} \frac{(-1)^i}{(m-i)!} ª^{m-i} \frac{«(m-i+1)}{(m-i+1)} = \frac{«_{(m+1)}}{(m+1)} \frac{(1-¨(ª))}{ª}.$$

Proof. Let

$$¤(ª) = \int_0^\infty\int_0^\infty e^{-ª(x+y)} dF(x,y).$$

Since F is BNBUL we have

$$\int_0^\infty\int_0^\infty e^{-ª(x+y)} \overline{F}(x+t, y+s) dxdy$$

$$\leq \overline{F}(t,s) \int_0^\infty\int_0^\infty e^{-ª(x+y)} \overline{F}(x,y) dydx$$

Multiplying both side by $(ts)^m$ and integrating over $(0,\infty)$, we obtain,

$$\int_0^\infty\int_0^\infty (ts)^m \int_0^\infty\int_0^\infty e^{-ª(x+y)} \overline{F}(x+t, y+s) dydxdsdt$$

$$\leq \int_0^\infty\int_0^\infty (ts)^m \overline{F}(t,s) dsdt \int_0^\infty\int_0^\infty e^{-ª(x+y)} \overline{F}(x,y) dydx \quad (3.19)$$

Consider

$$\int_0^\infty\int_0^\infty (ts)^m \overline{F}(t,s) dsdt = E\left[\int_0^\infty\int_0^\infty (ts)^m I(T > t, S > s) dsdt\right]$$

$$= \frac{«(m+1)}{(m+1)} \quad (3.20)$$

$$\int_0^\infty\int_0^\infty e^{-ª(x+y)} \overline{F}(x,y) dydx = \int_0^\infty\int_0^\infty e^{-ª(x+y)} (1 - F(x,y)) dydx$$

$$= \frac{1}{ª} - \int_0^\infty\int_0^\infty e^{-ª(x+y)} F(x,y) dydx$$

$$= \frac{1}{ª} - \frac{1}{ª} ¤(ª)$$

$$= \frac{1}{ª}(1 - ¤(ª)) \quad (3.21)$$

Using (3.20) and (3.21) in the equation (3.19) we have

$$= \frac{«_{(m+1)}}{m+1} \frac{1}{ª}[1 - ¤(ª)] \quad (3.22)$$

$$= \int_0^\infty\int_0^\infty (ts)^m \int_0^\infty\int_0^\infty e^{-ª(x+y)} \overline{F}(x+t, y+s) dydxdsdt$$

$$= \int_0^\infty\int_0^\infty e^{-ª(t+s)} \overline{F}(t,s) \int_0^s\int_0^t (u+v)^m e^{ª(u+v)} dvdudsdt \quad (3.23)$$

Consider

$$\int_0^s\int_0^t (u+v)^m e^{ª(u+v)} dvdu = \frac{m}{ª^{m+1}}\left[(-1)^{m+1} + \sum_{i=0}^{m} (-1)^i \frac{(ª(s+t))^m}{(m-i)!} e^{ª(s+t)}\right] \quad (3.24)$$

Therefore (3.23) becomes

$$= \int_0^\infty\int_0^\infty e^{-ª(t+s)} \overline{F}(t,s) \frac{m!}{ª^{m+1}} \left[(-1)^{m+1} + \sum_{i=0}^{m} (-1)^i \frac{[ª(t+s)]^{m-i}}{(m-i)!} e^{ª(s+t)}\right] dsdt$$

$$= \frac{(-1)^{m+1} m!}{ª^{m+1}}\left[\int_0^\infty\int_0^\infty e^{-ª(t+s)} \overline{F}(t,s) dsdt\right] + \frac{m!}{ª^{m+1}} \sum_{i=0}^{m} \frac{(-1)^i}{(m-i)!} \int_0^\infty\int_0^\infty ª^{m-i} (t+s)^{m-i} \overline{F}(t,s) dsdt$$

$$= \frac{(-1)^{m+1} m!}{ª^{m+1}}\left[1 - ¨(ª)\right] + \frac{m!}{ª^{m+1}} \sum_{i=0}^{m} \frac{(-1)^i}{(m-i)!} ª^{m-i} \frac{«(m-i+1)}{(m-i+1)} \quad (3.25)$$

Using (3.22) and (3.25) then the inequality (3.19) becomes

$$\frac{(-1)^{m+1} m!}{ª^{m+1}}\left[1 - ¨(ª)\right] + \frac{m!}{ª^{m+1}} \sum_{i=0}^{m} \frac{(-1)^i}{(m-i)!} ª^{m-i} \frac{«(m-i+1)}{(m-i+1)} = \frac{«_{(m+1)}}{(m+1)} \frac{(1-¨(ª))}{ª}$$

This completes the proof of the theorem.

Theorem 3.12 Let F be a life distribution which is BEBU with mean $«$ then

$$«^m ª_n \leq ª_{m+n}$$

Proof. We shall consider only the BEBU case

$$\overline{F}(x+t, y+s) \leq \overline{F}(t,s) e^{\frac{(x+y)}{«}}$$

Multiplying both sides by $\frac{(xy)^{m-1}(ts)^{n-1}}{¥(m)\Gamma(n)}$ and integrating over $(0,\infty)$, we obtain,

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{(xy)^{m-1}(ts)^{n-1}}{\Gamma(m)\Gamma(n)} \overline{F}(x+t, y+s) dxdydsdt$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{(xy)^{m-1}(ts)^{n-1}}{\Gamma(m)\Gamma(n)}\overline{F}(t,s)e^{\frac{(x+y)}{«}}\,dxdydsdt$$

(3.26)

$$= \left(\int_0^\infty\int_0^\infty \frac{(xy)^{m-1}}{\Gamma(m)}e^{\frac{(x+y)}{«}}\,dxdy\right)\left(\int_0^\infty\int_0^\infty \frac{(ts)^{m-1}}{\Gamma(n)}\overline{F}(t,s)dsdt\right)$$

$$= «^m\, \text{a}_n \qquad (3.27)$$

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{(xy)^{m-1}(ts)^{n-1}}{\Gamma(m)\Gamma(n)}\overline{F}(x+t,y+s)dxdydsdt$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \frac{(xy)^{m-1}(ts)^{n-1}}{\Gamma(m)\Gamma(n)}\int_{x+t}^\infty\int_{y+s}^\infty dF(u,v)dxdydsdt$$

On changing the variables we get,

$$= \int_0^\infty\int_0^\infty\left(\int_0^{z_1}\int_0^{z_2}\frac{(xy)^{m-1}}{\Gamma(m)}\right)\left(\int_0^{z_1-x}\int_0^{z_2-y}\frac{(ts)^{n-1}}{\Gamma(n)}dsdt\right)dF(z_1,z_2)$$

$$= \int_0^\infty\int_0^\infty\int_0^{z_1}\int_0^{z_2}\frac{(xy)^{m-1}}{\Gamma(m)}\frac{(z_1-x)^n(z_2-y)^n}{\Gamma(n+1)}\,dxdy\;dF(z_1,z_2)$$

$$= \int_0^\infty\int_0^\infty\int_0^1\int_0^1\frac{(\sim z_1.\;\; z_2)^{m-1}z_1^n(1-\sim)^n z_2^n(1-\;\;)^n}{\Gamma(m)\Gamma(n+1)}\,(z_1d\sim)(z_2d\;\;)dF(z_1,z_2)$$

$$= \int_0^\infty\int_0^\infty\int_0^1\int_0^1\frac{(z_1z_2)^{m-1}z_1^{n+1}z_2^{n+1}}{\Gamma(m)\Gamma(n+1)}(\sim\;\;)^{m-1}\,[(1-\sim)(1-\;\;)]^n d\sim d\;\; dF(z_1,z_2)$$

$$= \int_0^\infty\int_0^\infty\frac{z_1^{m+n}z_2^{m+n}}{\Gamma(m)\Gamma(n+1)}\left(\int_0^1\int_0^1(\sim\;\;)^{m-1}[(1-\sim)(1-\;\;)]^n d\sim d\;\;\right)dF(z_1,z_2)$$

$$= \int_0^\infty\int_0^\infty\frac{(z_1z_2)^{m+n}}{\Gamma(m)\Gamma(n+1)}\frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)}dF(z_1,z_2)$$

$$= \frac{1}{\Gamma(m+n+1)}\int_0^\infty\int_0^\infty(z_1z_2)^{m+n}dF(z_1,z_2)$$

$$= \frac{E\left[(z_1z_2)^{m+n}\right]}{\Gamma(m+n+1)} = \frac{«^{m+n}}{\Gamma(m+n+1)}$$

$$= \text{a}_{m+n}. \qquad (3.28)$$

Using (3.27) and (3.28) in (3.26) we get,

$$«^m\text{a}_n \le \text{a}_{m+n}$$

This completes the proof of the theorem.

Theorem 3.13 Suppose that F is BEBUC(2) life distribution such that its $«_{r+s+4}$ the moment of order is finite $(r+s+4)$ for some integers r and s then the following moment inequality holds

$$\frac{«_{m+n+3}}{(m+n+3)!} \le \frac{«_{(m+1)}«_{n+2}}{(n+2)!}$$

Proof. Since F is BEBUC

$$\int_u^\infty\int_v^\infty\overline{F}(x+t,y+s)dsdt \le e^{\frac{(x+y)}{«}}\int_u^\infty\int_v^\infty\overline{F}(t,s)dsdt \qquad (3.29)$$

Multiplying both sides of (3.29) by $(xy)^m(ts)^n$ and integrating we get

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_u^\infty\int_v^\infty(xy)^m(ts)^n\overline{F}(x+t,y+s)dxdydsdtdvdu$$

$$\le \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{x+y}{«}\right)}(xy)^m(ts)^n\int_u^\infty\int_v^\infty\overline{F}(t,s)dsdtdvdudxdy \qquad (3.30)$$

Also

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-\left(\frac{x+y}{«}\right)}(xy)^m(ts)^n\int_u^\infty\int_v^\infty\overline{F}(t,s)dsdtdvdudxdy$$

$$= \int_0^\infty\int_0^\infty(xy)^m e^{-\left(\frac{x+y}{«}\right)}\int_0^\infty\int_0^\infty(ts)^n\int_u^\infty\int_v^\infty\overline{F}(t,s)dsdtdvdudydx$$

$$= m!«^{m+1}\int_0^\infty\int_0^\infty\overline{F}(ts)\int_0^u\int_0^v(t,s)^n dtdsdudv$$

$$= \frac{m!«^{m+1}}{(n+1)}\int_0^\infty\int_0^\infty(uv)^{n+1}\overline{F}(t,s)dsdt$$

$$= \frac{m!«^{m+1}«^{n+2}}{(n+1)(n+2)} \qquad (3.31)$$

Also

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_u^\infty\int_v^\infty(xy)^m(ts)^n\overline{F}(x+t,y+s)\,dxdydsdtdvdu$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_u^\infty\int_v^\infty(xy)^m(ts)^n\overline{F}(x+t,y+s)\,dxdydsdtdvdu$$

$$= \frac{m!n!«_{(m+n+3)}}{(m+n+3)!} \qquad (3.32)$$

Using (3.31) and (3.32) in (3.30) we have

$$\frac{m!n!«_{(m+n+3)}}{(m+n+3)!} \le \frac{m!«_{(m+1)}«_{(n+2)}}{(n+1)(n+2)}$$

$$\frac{«_{(m+n+3)}}{(m+n+3)!} \le \frac{«_{(m+1)}«_{(n+2)}}{(n+2)!}$$

This completes the proof of the theorem.

Theorem 3.14 If F is BEBUCA then

$$\frac{«_{(m+3)}}{(m+1)(m+2)(m+3)} \le «^2«_{(m+1)}$$

Proof. Since F is BEBUCA we have

$$\int_0^\infty\int_0^\infty\overline{W}(x+t,y+s)dxdy \le «^2\overline{F}(t,s).$$

Multiplying both sides by $(ts)^m$ and integrating we get

$$\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty (ts)^m \overline{W}(x+t,y+s)dxdydtds \qquad (3.33)$$
$$\le «^2 \int_0^\infty\int_0^\infty (ts)^m \overline{F}(t,s)dsdt$$

$$= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty (ts)^m \overline{W}(x+t,y+s)dxdydtds$$
$$= \int_0^\infty\int_0^\infty\int_0^{z_2}\int_0^{z_1} \overline{W}(z_1 z_2)(ts)^m\, dtdsdz_1 dz_2$$
$$= \int_0^\infty\int_0^\infty \overline{W}(z_1 z_2)\frac{t^{m+1}}{m+1}\Big|_0^{z_1} \frac{s^{m+1}}{m+1}\Big|_0^{z_2}\, dz_1 dz_2$$
$$= \int_0^\infty\int_0^\infty \overline{W}(z_1 z_2)\frac{(z_1 z_2)^{m+1}}{(m+1)^2} dz_1 dz_2$$
$$= \frac{1}{(m+1)^2}\int_0^\infty\int_0^\infty (z_1 z_2)^{m+1}\overline{W}(z_1 z_2)dz_1 dz_2$$
$$= \frac{1}{(m+1)^2}\int_0^\infty\int_0^\infty (xy)^{m+1}\overline{W}(xy)dxdy$$
$$= \frac{1}{(m+1)^2} E\left[\int_0^\infty\int_0^\infty (xy)^{m+1}(XY-xy)I(X>x,Y>y)dydx\right]$$
$$= \frac{1}{(m+1)^2} E\left[XY\int_0^X\int_0^Y (xy)^{m+1}dydx - \int_0^X\int_0^Y (xy)^{m+2}dydx\right]$$
$$= \frac{1}{(m+1)^2} E\left[\frac{(XY)^{m+3}}{(m+1)^2}\left(\frac{1}{m+2}-\frac{1}{m+3}\right)\right]$$
$$= \frac{«_{(m+3)}}{(m+1)^2(m+2)(m+3)} \qquad (3.34)$$
$$= «^2 \int_0^\infty\int_0^\infty (ts)^m \overline{F}(t,s)dsdt$$
$$= «^2 \frac{«_{(m+1)}}{(m+1)} \qquad (3.35)$$

Using (3.34) and (3.35) in (3.33) we get

$$\frac{«_{(m+3)}}{(m+1)^2(m+2)(m+3)} \le «^2 \frac{«_{(m+1)}}{(m+1)}$$

$$\frac{«_{(m+3)}}{(m+1)(m+2)(m+3)} \le «^2 «_{(m+1)}$$

This completes the proof of the theorem.

Conclusion

In this paper, we have derived the moment inequalities for bivariate ageing classes of life distributions.

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