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Research Article

FIXED POINT THEOREMS WITH DIGITAL CONTRACTIONS

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A B S T R A C T

In this paper, we introduce some new Digital Contractions. Then, we state and prove fixed point theorems for digital images. Finally we deal with an application of these fixed point theorems to image processing.

INTRODUCTION

There are various applications of fixed point theory in mathematics, computer science, engineering, game theory, image processing, etc. Banach fixed point theorem is the most significant test for solution of some problems in mathematics and engineering. The dawn of the fixed point theory starts when in 1912 Brouwer [1] proved a fixed point result for continuous self maps on a closed ball. In 1922, Banach [2] gave a very useful result known as the Banach Contraction Principle. Kannan [3,4], then relaxed the condition of continuity of the map considered in Banach Contraction Principle in his paper in 1968. Zamfirescu [5] and Rhoades[6], consequently developed more general contractions for a complete metric space. These contractions have been generalised to the other spaces also by various authors [7-10].
Digital topology is a developing area which is related to features of 2D and 3D digital images using general topology and functional analysis. Up to now, several developments have been occurred in the study of digital topology. Digital topology was first studied by Rosenfield[11]. Kong [12], then introduced the digital fundamental group of a discrete object. Boxer [13] has given the digital versions of several notions from topology and [14] studied a variety of digital continuous functions. Some results and characteristic properties on the digital homology groups of 2D digital images are given in [15] and [16]. Ege and Karaca [17,18] give relative and reduced Lefschetz fixed point theorem for digital images.

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They also calculate degree of antipodal map for the sphere like digital images using fixed point properties. Ege and Karaca [19] then defined a digital metric space and proved the famous Banach Contraction Principle for digital images.

Preliminaries

Let X be a subset of $\mathbb{Z}^n$ for a positive integer n where $\mathbb{Z}^n$ is the set of lattice points in the n- dimensional Euclidean space and ρ represent an adjacency relation for the members of X. A digital image consists of (X, ρ).

Definition [20]: Let l, n be positive integers, $1 \leq l \leq n$ and two distinct points

$a = (a_1, a_2, \dots, a_n), b = (b_1, b_2, \dots, b_n) \in \mathbb{Z}^n$
a and b are k_l- adjacent if there are at most l indices i such that $|a_i - b_i| = 1$ and for all other indices j such that $|a_j - b_j| \neq 1$, $a_j = b_j$.

There are some statements which can be obtained from definition 2.1:

- a and b are 2- adjacent if $|a - b| = 1$.
- a and b in $\mathbb{Z}^2$ are 8- adjacent if they are distinct and differ by at most 1 in each coordinate.
- a and b in $\mathbb{Z}^3$ are 26- adjacent if they are distinct and differ at most 1 in each coordinate.
- a and b in $\mathbb{Z}^3$ are 18- adjacent if are 26- adjacent and differ by at most two coordinates.
- a and b are 6- adjacent if they are 18- adjacent and differ in exactly one coordinate.

A ρ-neighbour [20] of $a \in \mathbb{Z}^n$ is a point of $\mathbb{Z}^n$ that is ρ-adjacent to a where $\rho \in \{2,4,8,6,18,26\}$ and $n \in 1,2,3$. The set

$N_\rho(a) = \{b | b \text{ is } \rho - \text{adjacent to } a\}$

is called the ρ- neighbourhood of a. A digital interval [10] is defined by

$[p, q]_{\mathbb{Z}} = \{z \in \mathbb{Z} | p \leq z \leq q\}$

where $p, q \in \mathbb{Z}$ and $p < q$.

A digital image $X \subset \mathbb{Z}^n$ is ρ- connected [21] if and only if for every pair of different points $u, v \in X$, there is a set $\{u_0, u_1, \dots, u_r\}$ of points of digital image X such that $u = u_0, v = u_r$ and u_i and u_{i+1} are ρ- neighbours where $i = 0,1, \dots, r-1$.

Definition: Let $(X, \rho_0) \subset \mathbb{Z}^{n_0}$, $(Y, \rho_1) \subset \mathbb{Z}^{n_1}$ be digital images and $T: X \to Y$ be a function.

- T is said to be (ρ_0, ρ_1)- continuous[20], if for all ρ_0- connected subset E of X, $f(E)$ is a ρ_1- connected subset of Y.
- For all ρ_0- adjacent points $\{u_0, u_1\}$ of X, either $T(u_0) = T(u_1)$ or $T(u_0)$ and $T(u_1)$ are a ρ_1- adjacent in Y if and only if T is (ρ_0, ρ_1)- continuous [20].
- If f is (ρ_0, ρ_1)- continuous, bijective and T^{-1} is (ρ_1, ρ_0)- continuous, then T is called (ρ_0, ρ_1)- isomorphism [22]and denoted by $X \cong_{(\rho_0, \rho_1)} Y$.

A $(2, \rho)$- continuous function T, is called a digital ρ- path [20] from u to v in a digital image X if $T: [0, m]_{\mathbb{Z}} \to X$ such that $T(0) = u$ and $T(m) = v$. A simple closed ρ- curve of $m \geq 4$ points [23] in a digital image X is a sequence $\{T(0), T(1), \dots, T(m-1)\}$ of images of the ρ- path $T: [0, m-1]_{\mathbb{Z}} \to X$ such that $T(i)$ and $T(j)$ are ρ- adjacent if and only if $j = i \pm mod\ m$.

Definition: [19]: A sequence $\{x_n\}$ of points of a digital metric space (X, d, ρ) is a Cauchy sequence if for all $\in > 0$, there exists $\delta \in \mathbb{N}$ such that for all $n, m > \delta$, then

$$d(x_n, x_m) < \in.$$

Definition:[19]: A sequence $\{x_n\}$ of points of a digital metric space (X, d, ρ) converges to a limit $p \in X$ if for all $\in > 0$, there exists $\alpha \in \mathbb{N}$ such that for all $n > \delta$, then

$$d(x_n, p) < \in.$$

Definition: [19]: A digital metric space (X, d, ρ) is a digital metric space if any Cauchy sequence $\{x_n\}$ of points of (X, d, ρ) converges to a point p of (X, d, ρ).

Definition: [19]: Let (X, ρ) be any digital image. A function $T: (X, \rho) \to (X, \rho)$ is called right- continuous if $f(p) = \lim_{x \to p^+} T(x)$ where, $p \in X$.

Definition: [19]: Let, (X, d, ρ) be any digital metric space and $T: (X, d, \rho) \to (X, d, \rho)$ be a self digital map. If there exists $\alpha \in (0,1)$ such that for all $x, y \in X$,

$d(f(x), f(y)) \leq \alpha d(x, y)$,

then T is called a digital contraction map.

Proposition: [19]: Every digital contraction map is digitally continuous.

Theorem: [19]: (Banach Contraction principle) Let (X, d, ρ) be a complete metric space which has a usual Euclidean metric in $\mathbb{Z}^n$. Let, $T: X \to X$ be a digital contraction map. Then T has a unique fixed point, i.e. there exists a unique $p \in X$ such that $f(p) = p$.

Main results

***Definition*:** Let, (X, d, ρ) be any digital metric space and $T: (X, d, \rho) \to (X, d, \rho)$ be a self digital map. If there exists $\lambda \in \left(0, \frac{1}{2}\right)$ such that for all $x, y \in X$,

$$d(T(x), T(y)) \leq \lambda \Big(d(x, T(x)) + d(y, T(y))\Big),$$

then T is called a Kannan digital contraction map.

***Definition**:* Let, (X, d, ρ) be any digital metric space and $T: (X, d, \rho) \to (X, d, \rho)$ be a self digital map. If there exists $\lambda \in (0,1)$ such that for all $x, y \in X$,

$$d(T(x), T(y)) \leq \lambda \max\left\{d(x, y), \frac{\{d(x, T(x)) + d(y, T(y))\}}{2}, \frac{\{d(x, T(y)) + d(y, T(x))\}}{2}\right\},$$

then T is called a Zamfirescu digital contraction.

***Definition**:* Let, (X, d, ρ) be any digital metric space and $T: (X, d, \rho) \to (X, d, \rho)$ be a self digital map. If there exists $\lambda \in (0,1)$ such that for all $x, y \in X$,

$$d(T(x), T(y)) \leq \lambda \max\left\{d(x, y), \frac{\{d(x, T(x)) + d(y, T(y))\}}{2}, d(x, T(y)), d(y, T(x))\right\},$$

then T is called a Rhoades digital contraction.

Theorem: **(Kannan Contraction principle)** Let (X, d, ρ) be a complete metric space which has a usual Euclidean metric in $\mathbb{Z}^n$. Let, $T: X \to X$ be a Kannan digital contraction map. Then T has a unique fixed point, i.e. there exists a unique $c \in X$ such that $T(c) = c$.

***Proof**:* Let x_0 be any point of X. Consider the iterate sequence $T(x_n) = x_{n+1}$. Using induction on n, we obtain

$$\begin{aligned} d(x_{n+1}, x_n) &= d(T(x_n), T(x_{n-1})) \\ &\leq \lambda[d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2})] \\ &\leq 2\lambda d(x_n, x_{n-1}) \end{aligned}$$

Again

$$\begin{aligned} d(x_{n+1}, x_n) &\leq 2\lambda d(x_n, x_{n-1}) \leq (2\lambda)^2 d(x_{n-1}, x_{n-2}) \leq \cdots \\ &\leq (2\lambda)^n d(x_1, x_0) = (2\lambda)^n d(T(x_0), x_0) \end{aligned}$$

For natural numbers $n \in \mathbb{N}$ and $m \geq 1$, we conclude that

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, x_{n+m-1}) + \cdots + d(x_{n+1}, x_n) \\ &\leq [(2\lambda)^{n+m} + \cdots + (2\lambda)^n] d(T(x_0), x_0) \\ &\leq \frac{(2\lambda)^n}{1 - 2\lambda} d(f(x_0), x_0) \end{aligned}$$

As a result, x_n is a Cauchy sequence. There is a limit point of x_n because (X, d, ρ) is digital metric space. Let c be the limit of x_n. From the (ρ, ρ)- continuity of T, we get

$$T(c) = \lim_{n \to \infty} T(x_n) = \lim_{n \to \infty} x_{n+1} = c.$$

For uniqueness,

Assume that $a, b \in X$ are fixed points of f. Then we have the following:

$$\begin{aligned} d(a, b) = d(T(a), T(b)) &\leq \lambda[d(a, T(a)) + d(b, T(b))] \\ &= \lambda[d(a, a) + d(b, b)] \\ &\Rightarrow d(a, b) \leq 0 \Rightarrow a = b. \end{aligned}$$

Therefore, T has a unique fixed point.

***Remark**:* Every Digital Contraction in [17] is a Kannan Digital Contraction, but the converse is not true.

***Example**:* Let $X = [0,2]_{\mathbb{Z}}$ be a digital interval with 2-adjacency. Let, $d(x, y) = \min\{x, y\}$ be the digital metric. Consider the map $T: X \to X$ defined by

$$T(x) = \begin{cases} \frac{x}{2}, & if\ 0 \leq x \leq 1 \\ 1 + \frac{x}{2}, & if\ 1 < x \leq 2 \end{cases}$$

It is clear that T has a unique fixed point i.e., $x = 0$.
The given map is not a Digital Contraction but is clearly a Kannan Digital Contraction.

***Theorem:* (A generalization of Kannan Contraction principle)** Let (X, d, ρ) be a complete metric space which has a usual Euclidean metric d in $\mathbb{Z}^n$ and let $T: X \to X$ be a digital self map. Assume that there exists a right continuous real function

$\Upsilon: [0, u] \to \left[0, \frac{u}{2}\right]$

Where u is sufficiently large real number such that

$$\Upsilon(a) < \frac{a}{2}\ if\ a > 0, \tag{1}$$

And let f satisfies

$$d\big(T(x_1), T(x_2)\big) \leq \Upsilon\Big(d\big(x_1, T(x_1)\big) + d\big(x_2, T(x_2)\big)\Big) \tag{2}$$

For all $x_1, x_2 \in (X, d, \rho)$. Then T has a unique fixed point $c \in (X, d, \rho)$ and the sequence $T^n(x)$ converges to c for every $x \in X$.

Proof: We first prove the uniqueness. Let u_1, u_2 be two fixed points of T. By (3.1) and (3.2), we get $(u_1, u_2) = d\big(T(u_1), T(u_2)\big) \leq \Upsilon\big(d(u_1, u_1) + d(u_2, u_2)\big) \Rightarrow u_1 = u_2$.
Now, let's prove the existence. For this purpose, we take a point $x_0 \in (X, d, \rho)$ and define the sequence $T(x_n) = x_{n+1}$. For,$n \in \mathbb{N}$, define the following sequence:

$$a_n = d(x_n, x_{n-1}).$$

Using (3.1) and (3.2), we obtain

$$a_{n+1} = d(x_{n+1}, x_n) \leq \Upsilon\big(d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2})\big) < 2d(x_n, x_{n-1}) = 2a_n$$

For all $n \in \mathbb{N}$. Thus the sequence a_n is decreasing and so it has a limit a. If we assume that $a > 0$, we have

$$a_{n+1} \leq \Upsilon(a_n)$$

From (3.2). Since Υ is right continuous, we get

$$a \leq \Upsilon(a)$$

But this contradicts with (3.1). As a result, $a_n \to 0$ as $n \to \infty$.
We would like to prove x_n is a Cauchy sequence. Suppose x_n is not a Cauchy sequence. Then there exists $\epsilon > 0$ and integers $m > n > k$ for every $k \geq 1$ such that

$$d(x_m, x_n) \geq \epsilon.$$

For a smallest value of m, we can suppose that $d(x_{m-1}, x_n) < \epsilon$. If we use the triangle inequality, we obtain

$$\epsilon \leq d(x_m, x_n) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_n) < \epsilon + d(x_m, x_{m-1}).$$

Since $d(x_n, x_{n-1}) \to 0$ as $n \to \infty$.
From the fact that

$$m > n \Rightarrow d(x_{m+1}, x_m) \leq d(x_{n+1}, x_n)\ and\ d(x_m, x_{m-1}) \leq d(x_n, x_{n-1})$$

And (3.2), we have

$$\begin{aligned}\epsilon \leq d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{n+1}) + d(x_{n+1}, x_n) \\ &\leq 2d(x_{n+1}, x_n) + \Upsilon\big(d(x_m, x_{m-1}) + d(x_n, x_{n-1})\big) \\ &\leq 2d(x_{n+1}, x_n) + \Upsilon\big(2d(x_n, x_{n-1})\big)\end{aligned}$$

Taking the limit $n \to \infty$, from these inequalities we get $\epsilon \leq \Upsilon(2\epsilon)$ but this contradicts with (3.1) because $\epsilon > 0$. As a result, x_n is a Cauchy sequence and since (X, d, ρ) is a complete metric space, $f^n(x)$ converges in (X, d, ρ).

Remark: The Digital Contraction Map defined in [17] can be used in compressing a digital image for a convenient storing or other compressing requirements. But that contraction ensures the continuity of a function. Now, the Kannan Digital Contraction map defined in this paper need not require the continuity of the map, i.e. if we want to contract more than one image in a given array in a storage then this can be helpful. Moreover as a video is nothing but a collection of a finite number of digital images this contraction can also work for the video storage or compression.

***Theorem:* (Zamfirescu Contraction principle)** Let (X, d, ρ) be a complete metric space which has a usual Euclidean metric in $\mathbb{Z}^n$. Let, $f: X \to X$ be a Zamfirescu digital contraction. Then T has a unique fixed point, i.e. there exists a unique $c \in X$ such that $T(c) = c$.

Proof: Let x_0 be any point of X. Consider the iterate sequence $T(x_n) = x_{n+1}$. Using induction on n, we obtain

Case1: Let $max\left\{d(x, y), \frac{\{d(x,T(x))+d(y,T(y))\}}{2}, \frac{\{d(x,T(y))+d(y,T(x))\}}{2}\right\} = d(x, y)$

$$d(x_{n+1}, x_n) = d\big(T(x_n), T(x_{n-1})\big) \leq \lambda d(x_n, x_{n-1}) \leq \cdots \leq \lambda^n d(T(x_0), x_0)$$

Case 2: Let $max\left\{d(x, y), \frac{\{d(x,T(x))+d(y,T(y))\}}{2}, \frac{\{d(x,T(y))+d(y,T(x))\}}{2}\right\} = \frac{\{d(x,T(x))+d(y,T(y))\}}{2}$

$$\begin{aligned} d(x_{n+1}, x_n) &= d\big(T(x_n), T(x_{n-1})\big) \\ &\leq \lambda \frac{[d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2})]}{2} \\ &\leq \lambda d(x_n, x_{n-1})\end{aligned}$$

Again

$$d(x_{n+1}, x_n) \leq \lambda d(x_n, x_{n-1}) \leq (\lambda)^2 d(x_{n-1}, x_{n-2}) \leq \cdots \leq (\lambda)^n d(x_1, x_0) = (\lambda)^n d(f(x_0), x_0)$$

Case 3: Let $max\left\{d(x, y), \frac{\{d(x,T(x))+d(y,T(y))\}}{2}, \frac{\{d(x,T(y))+d(y,T(x))\}}{2}\right\} = \frac{\{d(x,T(y))+d(y,T(x))\}}{2}$

$$\begin{aligned} d(x_{n+1}, x_n) &= d\big(f(x_n), f(x_{n-1})\big) \\ &\leq \lambda \frac{\{d(x_n, x_{n-2}) + d(x_{n-1}, x_{n-1})\}}{2} \\ &= \frac{\lambda}{2} d(x_n, x_{n-2}) \leq \lambda d(x_n, x_{n-1})\end{aligned}$$

Again $d(x_{n+1}, x_n) = d\big(T(x_n), T(x_{n-1})\big) \leq \lambda d(x_n, x_{n-1}) \leq \cdots \leq \lambda^n d(T(x_0), x_0)$

For natural numbers $n \in \mathbb{N}$ and $m \geq 1$, we conclude that

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, x_{n+m-1}) + \cdots + d(x_{n+1}, x_n) \\ &\leq [(\lambda)^{n+m} + \cdots + (\lambda)^n] d(T(x_0), x_0) \\ &\leq \frac{(\lambda)^n}{1 - \lambda} d(T(x_0), x_0)\end{aligned}$$

As a result, x_n is a Cauchy sequence. There is a limit point of x_n because (X, d, ρ) is digital metric space. Let c be the limit of x_n. From the (ρ, ρ)- continuity of T, we get

$$T(c) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = c.$$

For uniqueness,
Assume that $a, b \in X$ are fixed points of T. Then we have the following:

$$d(a,b) = d\big(T(a),T(b)\big)$$
$$\le \lambda max\left\{d(a,b), \frac{\{d(a,T(a)) + d(b,T(b))\}}{2}, \frac{\{d(a,T(b)) + d(b,T(a))\}}{2}\right\}$$
$$\le \lambda d(a,b)$$
$$\Rightarrow d(a,b) \le 0 \;\Rightarrow a = b.$$

Therefore, T has a unique fixed point.

Theorem: **(Rhoades Contraction principle)** Let (X,d,ρ) be a complete metric space which has a usual Euclidean metric in $\mathbb{Z}^n$. Let, $T: X \to X$ be a Rhoades digital contraction map. Then T has a unique fixed point, i.e. there exists a unique $c \in X$ such that $T(c) = c$.

P*roof*: Let x_0 be any point of X. Consider the iterate sequence $T(x_n) = x_{n+1}$. Using induction on n, we obtain

$$d(x_{n+1},x_n) = d(T(x_n),T(x_{n-1}))$$
$$\le \lambda max\left\{d(x_n,x_{n-1}), \frac{\{d(x_n,T(x_n)) + d(x_{n-1},T(x_{n-1}))\}}{2}, d(x_n,T(x_{n-1})), d(x_{n-1},T(x_n))\right\}$$

Case1: Let

$$max\left\{d(x_n,x_{n-1}), \frac{\{d(x_n,T(x_n)) + d(x_{n-1},T(x_{n-1}))\}}{2}, d(x_n,T(x_{n-1})), d(x_{n-1},T(x_n))\right\} = d(x_n,x_{n-1})$$

Then,

$$d(x_{n+1},x_n) = d\big(T(x_n),T(x_{n-1})\big) \le \lambda d(x_n,x_{n-1}) \le \cdots$$
$$\le \lambda^n d(T(x_0),x_0)$$

Case2: Let

$$max\left\{d(x_n,x_{n-1}), \frac{\{d(x_n,T(x_n)) + d(x_{n-1},T(x_{n-1}))\}}{2}, d(x_n,T(x_{n-1})), d(x_{n-1},T(x_n))\right\}$$
$$= \frac{\{d(x_n,T(x_n)) + d(x_{n-1},T(x_{n-1}))\}}{2}$$

Then,

$$d(x_{n+1},x_n) = d\big(T(x_n),T(x_{n-1})\big)$$
$$\le \lambda \frac{[d(x_n,x_{n-1}) + d(x_{n-1},x_{n-2})]}{2}$$
$$\le \lambda d(x_n,x_{n-1})$$

Again

$$d(x_{n+1},x_n) \le \lambda d(x_n,x_{n-1}) \le (\lambda)^2 d(x_{n-1},x_{n-2}) \le \cdots$$
$$\le (\lambda)^n d(x_1,x_0) = (\lambda)^n d(T(x_0),x_0)$$

Case3: Let

$$max\left\{d(x_n,x_{n-1}), \frac{\{d(x_n,T(x_n)) + d(x_{n-1},T(x_{n-1}))\}}{2}, d(x_n,T(x_{n-1})), d(x_{n-1},T(x_n))\right\}$$
$$= d(x_n,T(x_{n-1}))$$

Then,

$$d(x_{n+1},x_n) = d\big(T(x_n),T(x_{n-1})\big) \le \lambda d\big(x_n,T(x_{n-1})\big)$$
$$\le \lambda d(x_n,x_{n-1})$$

Again,

$$d(x_{n+1},x_n) = d\big(T(x_n),T(x_{n-1})\big) \le \lambda d(x_n,x_{n-1}) \le \cdots$$
$$\le \lambda^n d(T(x_0),x_0)$$

Case4: Let

$$max\left\{d(x_n,x_{n-1}), \frac{\{d(x_n,T(x_n)) + d(x_{n-1},T(x_{n-1}))\}}{2}, d(x_n,T(x_{n-1})), d(x_{n-1},T(x_n))\right\}$$
$$= d(x_{n-1},T(x_n))$$

Now, $d\left(x_{n-1},T(x_n)\right) = d(x_{n-1},x_{n-1}) = 0$

So, $d(x_{n+1},x_n) = 0 \le \lambda d(x_n,x_{n-1})$

Again,

$$d(x_{n+1},x_n) = d\big(T(x_n),T(x_{n-1})\big) \le \lambda d(x_n,x_{n-1}) \le \cdots$$
$$\le \lambda^n d(T(x_0),x_0)$$

For natural numbers $n \in \mathbb{N}$ and $m \ge 1$, we conclude that

$$d(x_{n+m},x_n) \le d(x_{n+m},x_{n+m-1}) + \cdots + d(x_{n+1},x_n)$$
$$\le [(\lambda)^{n+m} + \cdots + (\lambda)^n] d(T(x_0),x_0)$$
$$\le \frac{(\lambda)^n}{1-\lambda} d(T(x_0),x_0)$$

As a result, x_n is a Cauchy sequence. There is a limit point of x_n because (X,d,ρ) is digital metric space. Let c be the limit of x_n. From the (ρ,ρ)- continuity of T, we get

$$T(c) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = c.$$

For uniqueness,

Assume that $a,b \in X$ are fixed points of T. Then we have the following:

$$d(a,b) = d\big(T(a),T(b)\big)$$
$$\le \lambda max\left\{d(a,b), \frac{\{d(a,T(a)) + d(b,T(b))\}}{2}, d(a,T(b)), d(b,T(a))\right\}$$
$$\le \lambda d(a,b)$$
$$\Rightarrow d(a,b) \le 0 \;\Rightarrow a = b.$$

Therefore, T has a unique fixed point.

Application of Kannan Fixed Point Theorem to image processing

In this section, we give an application of contraction principles to the image compression. The aim of image compression is to reduce a redundant image to digital image. There are some problems in storing an image. Memory data is usually too large and stored image does not have enough information to the original one. Due to this fact the quality of image stored can be poor. For this one can use the method developed by Barnsley, iterated function system. The procedure is as follows and depends on the contraction principles:

- Consider a triangle.
- Make three copies of it, which is exactly half of it in dimension.
- Paste one copy such that the top vertices of these triangles coincide.
- Paste the second copy to the left vertex of the triangle.
- Paste the last copy to the right vertex.
- Now again we have three triangles inside one triangle.
- Make 3 copies of each in the same manner of half the dimensions and paste accordingly.
- Repeat the above steps again and again.
- In this manner we get a nested triangle structure which is called The Serpin´ski carpet.

This was the contraction procedure of a single image. If we want to store two different images such as a triangular nested structure and a square nested structure. In this case we define two portions of the function, the first one will reduce the triangle to the half dimensions and the second portion of the function will reduce the square to half its dimensions. And repeating the above steps we get the required result.

From the above discussions one can easily conclude that as the Banach contraction principle is useful for the storage of continuous digital images, Kannan contraction ensures that it can be used to store discontinuous images also, or in other words more than one image can be contracted to save in the same slot. The authors also propose that with some conditions the same can may be used to even store a digital video.

Scope of the study

The aim of this paper is to give the digital version of some fixed point theorems where function need not be continuous. We hope the results will be useful in digital topology and fixed point theory. These results can therefore be used for the contraction of digital images and the array of digital images. In future, some other properties of digital images can be discussed with the viewpoint of fixed point theory.

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