International Journal of Current Advanced Research
ISSN: O: 2319-6475, ISSN: P: 2319-6505, Impact Factor: SJIF: 5.995
Available Online at www.journalijcar.org
Volume 7; Issue 1(H); January 2018; Page No. 9267-9270
DOI: http://dx.doi.org/10.24327/ijcar.2018.9270.1526

Research Article

SOME STABILITY CRITERION FOR THE SOLUTIONS OF FIRST ORDER DIFFERENCE EQUATION

P. U. Chopade*

Department of Mathematics, Dnyanopasak College, Jintur-431 509, India

A R T I C L E I N F O

Article History:
Received 20th October, 2017
Received in revised form 10th November, 2017
Accepted 26th December, 2017
Published online 28th January, 2018

Key words:
Difference equation, Equistability, Uniformly stable, Maximal solution.

A B S T R A C T

In this paper, we present some stability criterion for the solutions of first order difference equation applying various conditions.

INTRODUCTION

In the recent years the theory and applications of difference equations are found to be more useful in the engineering field. Agarwal [1], Kelley and Peterson [12] developed the theory of difference equations and difference inequalities. Existence of solutions for some summation equations are obtained by K. L. Bondar, A. B. Jadhav and M. R. Pawade [10]. K. L. Bondar and M. R. Pawade studied some summation inequalities reducible to difference inequalities are given in [4]. Some differential and integral inequalities are given in [13]. K. L. Bondar contributed δ -approximate solution of summation equation in [8, 9]. K. L. Bondar, V. C. Borkar and S. T. Patil discussed some comparison results along with existence and uniqueness for the first order difference equation in [2, 3]. K. L. Bondar contributed some difference inequalities, solutions of summation equations and some summation inequalities in [5, 6, 7, 8, 9]. Some comparison results in difference equations are given by A. B. Jadhav, P. U. Chopade and K. L. Bondar in [11]. In this paper we present some stability criterion of solutions for the first order difference equation applying various conditions.

Definitions and Preliminary Notes

Consider the difference equation

$$\Delta x(t) = f(t,x), x(t_0) = x_0, t_0 \in J, \tag{2.1}$$

where $f \in C[J \times R, R_+], J = \{t_0, t_0+1, t_0+2, \ldots, t_0+a\}, t_0 \in R_+$, the set of all non-negative real numbers.

Corresponding author: **P. U. Chopade**
Department of Mathematics, Dnyanopasak College, Jintur-431 509, India

Definition 2.1

For $V \in C[J \times R, R_+]$, we define the function

$$\Delta^+ V(t,x) = \sup_{t \in J}[V(t+1, x+f(t,x)) - V(t,x)] \tag{2.2}$$

for $(t,x) \in J \times R$.

Definition 2.2

Let $r(t)$ be any solution of (2.1) on *J*. Then $r(t)$ is said to be maximal solution of (2.1), if every solution $x(t)$ of (2.1) existing on *J*, the inequality $x(t) \le r(t)$ holds for $t \in J$.

Let $x(t, t_0, x_0)$ be any solution of the difference equation

$$\Delta x(t) = f(t,x), x(t_0) = x_0, \quad t_0 \ge 0, \tag{2.3}$$

where $f \in C[J \times S_\rho, R_+]$, S_ρ being the set

$$S_\rho = \{x \in R, |x| < \rho\}. \tag{2.4}$$

Assume that $f(t,0) = 0, t \in J$, so that $x = 0$ is a trivial solution of (2.3) through $(t_0, 0)$. We list a few definitions concerning the stability of the trivial solution.

Definition 2.3

The trivial solution $x = 0$ of (2.3) is

(S_1) equistable if for each $\epsilon > 0, t_0 \in J$, there exists a positive function $\delta = \delta(t_0, \epsilon)$ that is continuous in t_0 for each ϵ such that the inequality

$$|x_0| \le \delta$$

implies

$$|x(t, t_0, x_0)| < \epsilon, t \ge t_0;$$

(S$_2$) uniformly stable if the δ in (S$_1$) is independent of t_0.

Remark 2.1

Clearly ϵ given in the preceding definition must be less than ρ of (2.4), and therefore the concepts (S$_1$) and (S$_2$) are of local nature. If, on the other hand, $\rho = \infty$, so that $S\rho = R$, the corresponding concepts of stability would be of global character.

It is convenient to introduce certain classes of monotone functions.

Definition 2.4

A function $\varphi(r)$ is said to belong to the class K if $\varphi \in C\,[[0,\rho), R_+]$, $\varphi(0) = 0$, and $\varphi(r)$ is strictly monotone increasing in r.

Definition 2.5

A function $V(t,x)$ with $V(t,0) = 0$ is said to be positive definite if there exists a function $\varphi(r) \in K$ such that the relation

$$V(t,x) \geq \varphi(|x|)$$

is satisfied for $(t,\ x) \in J \times S_\rho$.

Definition 2.6

A function $V(t,x) \geq 0$ is said to be decrescent if a function $\varphi(r) \in K$ exists such that

$$V(t,x) \leq \varphi(|x|), (t,\ x) \in J \times S_\rho.$$

To study the scalar difference equation

$$\Delta u(t) = g(t,\ u(t)), \quad u(t_0) = u_0 \geq 0, \quad t_0 \geq 0, \qquad (2.5)$$

where $g \in C[J \times R_{+}, R]$.We suppose that $g(t,0) \equiv 0$ so that $u = 0$ is a solution of (2.5) through $(t_0,\ 0)$. Furthermore, this assumption also implies that the solutions $u(t) = u(t, t_0,\ u_0)$ of (2.5) are non-negative for $t \geq t_0$ so as to assure that $g(t, u(t))$ is defined.

Corresponding to the stability definitions (S$_1$) and (S$_2$), we designate by (S$_1^*$) and (S$_2^*$) the concepts concerning the stability of the solution $u = 0$ of (2.5).

Definition 2.7

The trivial solution $u = 0$ of (2.5) is said to be

(S$_1^*$) equistable if, for each $\epsilon > 0$, $t_0 \in J$, there exists a positive function$\delta = \delta(t_0, \epsilon)$ that is continuous in t_0 for each ϵ such that

$$u(t, t_0, u_0) < \epsilon, \quad t \geq t_0,$$

provided

$$u_0 \leq \delta;$$

(S$_2^*$) uniformly stable if the δ in (S$_1^*$) is independent of t_0.

Author proved following theorem in [12] which is used to prove the main results.

Theorem 2.1 [12]

Let $V \in C[J \times R, R_+]$ and $V(t,x)$ be locally Lipschitzian in x. Assume that the function $\Delta^+ V(\,t,x)$ of (2.2) satisfies

$$\Delta^+ V(x,t) \leq g\big(t, V(t,x)\big), \quad (t,x) \in J \times R. \qquad (2.6)$$

where $g \in C[J \times R_+, R\,]$. Let $r(t) = r(t, t_0, u_0)$ be the maximal solution of the scalar difference equation

$$\Delta u(t) = g(t,u), \quad u(t_0) = u_0 \geq 0 \qquad (2.7)$$

existing to the right of t_0. If $x(t) = x(t, t_0, x_0)$ is any solution of (2.1) existing for $t \geq t_0$ such that

$$V(t_0, x_0) \leq u_0, \qquad (2.8)$$

then

$$V\big(t, x(t)\big) \leq r(t), \quad t \geq t_0.$$

Definition 2.8

A function $V \in C[J \times S_\rho, R_+]$ is said to be locally Lipschitzian in x, if for each $(t,x) \in J \times S_\rho$ there exists a constant $M > 0$ and $\delta_0 > 0$ such that $|x - x_0| < \delta_0$, implies

$$|V(t,x) - V(t,x_0)| \leq M|x - x_0|.$$

MAIN RESULTS

Theorem 3.1

Assume that there exist functions $V(t,x)$ and $g(t,u)$ satisfying the following conditions

(i) $g \in C[J \times R_+, R\,]$ and $g(t,0) \equiv 0$.
(ii) $V \in C[J \times S_\rho, R_+]$, $V(t,0) \equiv 0$ and $V(t,x)$is positive definite and locally Lipschitzian in x.
(iii) For $(t,x) \in J \times S_\rho$, $D^+ V(t,\ x\,) \leq g\big(t, V(t,\ x\,)\big)$.

Then the equistability of the trivial solution of (2.5) implies the equistability of the trivial solution of the difference equation (2.3).

Proof

By assumption, a function $b(r)$ of class *K* exists such that

$$V(t,x) \geq b(|x|), (t,x) \in J \times S_\rho. \qquad (3.1)$$

Let $0 < \epsilon < \rho$ and $t_0 \in J$ be given. Since the solution $u = 0$ is equistable, given $b(\epsilon) > 0$, $t_0 \in J$, there exists a positive function $\delta = \delta(t_0, \epsilon)$ that is continuous in t_0 for each ϵ, such that $u_0 \leq \delta$ implies

$$u(t, t_0, u_0) < b(\epsilon), \quad t \geq t_0. \qquad (3.2)$$

Choose $u_0 = V(t_0, x_0)$. Since $V(t,x)$ is continuous and $V(t,0) \equiv 0$, it is possible to find a positive function $\delta_1 = \delta_1(t_0,\epsilon)$ that is continuous in t_0 for each ϵ, satisfying the inequalities

$$|x_0| \leq \delta_1, \quad V(t_0, x_0) \leq \delta \qquad (3.3)$$

simultaneously. We claim that, if $|x_0| \leq \delta_1$,

$$|x(t, t_0, x_0)| < \epsilon, \quad t \geq t_0.$$

Suppose that this is not true. Then, there would exists a solution $x(t) = x(t, t_0, x_0)$ with $|x_0| \leq \delta_1$, and a $t_1 > t_0$ such that

$$|x(t_1)| = \epsilon, \quad |x(t)| \leq \epsilon, \quad t \in [t_0, t_1],$$

so that

$$V(t_1, x(t_1)) \geq b(\epsilon) \qquad (3.4)$$

because of (3.1). This means that $|x(t)| < \rho$ for $t \in [t_0, t_1]$, and hence the choice $u_0 = V(t_0, x_0)$ and condition (iii) give, as a consequence of Theorem 2.1, the estimate

$$V\big(t, x(t)\big) \leq r(t, t_0, u_0), \quad t \in [t_0, t_1], \qquad (3.5)$$

where $r(t, t_0, u_0)$ is the maximal solution of (2.5). The relations (3.2), (3.4) and (3.5) lead to the contradiction

$$b(\epsilon) \leq V\big(t_1, x(t_1)\big) \leq r(t_1, t_0, u_0) < b(\epsilon),$$

proving (S_1). The proof of the theorem is complete.

Theorem: 3.2

Under the assumption of Theorem 3.1, the uniform stability of the solution $u = 0$ of (2.5) *also implies the equistability of the trivial solution of* (2.3).

Proof

The proof follows from the proof of Theorem 3.1. In this case, although δ is independent of t_0, the relation (3.3) shows that δ_1 is not independent of t_0. Consequently, one gets only the equistability of the trivial solution of (2.3).

Corollary:3.1

Assume that there exists a function $V(t, x)$ verifying the following conditions

(i) $V \in C[J \times S_\rho, R_+]$, $V(t, 0) \equiv 0$ and $V(t, x)$ is positive definite and locally Lipschitzian in x.

(ii) $D^+V(t, x) \leq 0$, $(t, x) \in J \times S_\rho$.

Then, the trivial solution of (2.3) *is equistable.*

Proof

It is important to note that, when (ii) holds, the scalar difference equation (2.5) reduces to

$$\Delta u(t) = 0, \quad u(t_0) = u_0, \quad t_0 \geq 0,$$

and as a result (S_2^*) is satisfied. Thus Corollary 3.1 follows from Theorem 3.2.

Theorem: 3.3

In addition to the hypothesis of Theorem3.1, assume that $V(t, x)$ is decrescent. Then, the equistability of null solution of (2.5) *assures the equistability of the solution $x = 0$ of* (2.3).

Proof

Since $V(t, x)$ is decrescent, there exists a function $a(r) \in K$ such that

$$V(t, x) \leq a|x|, \qquad (t, x) \in J \times S_\rho.$$

We follow the proof of Theorem 3.1 except that we choose $u_0 = a|x_0|$. By assumption, (S_1^*) holds, and therefore $\delta = \delta(t_0, \epsilon)$ depends on t_0. As $a(r) \in K$, the existence of a positive function $\delta_1 = \delta_1(t_0, \epsilon)$ satisfiying the inequalities

$$|x_0| < \delta_1, \quad a|x| \leq \delta \tag{3.6}$$

simultaneously is clear. The rest of the proof is very much the same.

Theorem: 3.4

Let the hypothesis of Theorem 3.1 hold. Assume further that $V(t, x)$ is decrescent. Then the uniform stability of the solution u of (2.5) *guarantees the uniform stability of the trivial solution of* (2.3).

Proof

Following the proof of Theorem 3.3, it is easy to see that δ_1 does not depend on t_0. For, by assumption of the uniform stability of the null solution of (2.5), δ is independent of t_0, and (3.6) shows that δ_1 is also independent of t_0.

Corollary: 3.2

Assume that there exists a function $V(t, x)$ fulfilling the following assumptions

(i) $V \in C[J \times S_\rho, R_+]$, $V(t, x)$ is positive definite and descrescent and locally Lipschitzian in x.

(ii) $D^+V(t, x) \leq 0$, $(t, x) \in J \times S_\rho$.

Then, the trivial solution of (2.3) *is uniformly stable.*

The definition of uniformly stability of the solution $x = 0$ given in (S_2) can also be formulated by means of monotone function, as can be seen by the following

Theorem: 3.5

The trivial solution of (2.3) *is uniformly stable if and only if there exists a function $a(r) \in K$ verifying the estimate*

$$|x(t, t_0, u_0)| \leq a|x_0|, \quad t \geq t_0 \tag{3.7}$$

for $|x_0| < \rho$.

Proof

The sufficiency of the condition is immediately clear. To prove the necessity, consider, for a given $\epsilon > 0$, the least upper bound for all positive function $\delta(\epsilon)$, and designate it by $\delta = \delta(\epsilon)$. Then $|x_0| \leq \delta$ implies $|x(t, t_0, x_0)| \leq \epsilon$ for $t \geq t_0$, and, if $\delta_1 > \delta$, there exists at least one x_0 such that , for $|x_0| \leq \delta_1$, $|x(t, t_0, u_0)|$ exceeds the value ϵ at some time t. Clearly, the function $\delta(\epsilon)$ is positive for $\epsilon > 0$; it is nondecreasing and tends to zero as $\epsilon \to \infty$; and it may be discontinuous. We now choose a continuous, monotonically increasing function $\delta^*(\epsilon)$ satisfying $\delta^*(\epsilon) \leq \delta(\epsilon)$. Then, the inverse function

$$a(r) = (\delta^*)^{-1}(r)$$

satisfies (3.7). This completes the proof.

References

1. R. P. Agarwal, ``Difference equations and inequalities: Theory, Methods and Applications,'' Marcel Dekker, New York, (1991).
2. K. L. Bondar, S. T. Patil, V. C. Borkar, "Comparison Theorems for Linear Difference Equation", *The Mathematics Education,* Vol. XLIV, No. 4, Dec, 2010.
3. K. L. Bondar, S. T. Patil, V. C. Borkar, "Some Existence and Uniqueness Results for Difference Boundary Value Problems*", Bulletin of Pure and Applied Sciences,* Vol. 29, Issue-2(2010), p. 291-296.
4. K. L. Bondar and M. R. Pawade, "On Some summation inequalities", *Journal of Contemporary Applied Mathematics,* Vol. 2, No. 1, Sept, 2011.
5. K. L. Bondar, "On Minimax Solution of First order Difference Initial Value Problems", *Journal of Contemporary Applied Mathematics*, Vol. 1, No. 1, Sept, 2011.
6. K. L. Bondar, "Some Comparison Results for First Order Difference Equations", *Int. J. Contemp. Math. Science,* Vol. 6, 2011, No. 38, 1855-1860.
7. K. L. Bondar, "Some Scalar Difference Inequalities", *Applied Mathematical Sciences*, Vol. 5, 2011, no. 60, 2951-2956.
8. K. L. Bondar, "Some summation inequalities reducible to difference inequalities", *Int. J. Contemp. Math. Science*, Vol. 2, No. 1, June, 2011.

9. K. L. Bondar, "On Solutions of Summation Equation", *Int. J. of Emerging trends in Engineering and Development*, Issue 1, Vol. 3, 2011.
10. K. L. Bondar, A. B. Jadhav and M. R. Pawade, "Local and Global Existence of Solutions for Summation Equation", *Int. J. of Pure and Applied Sciences and Technology*, 13(1), (2012), pp. 1-5.
11. A. B. Jadhav, P. U. Chopade, K. L. Bondar, "Some comparison results in difference equations'', *Journal of global research in mathematical archives*, Vol. 4, No. 10, Oct, 2017.
12. W. Kelley and A. Peterson, "Difference Equations", Academic Press, (2001), California, USA.
13. V. Lakshmikantham and S. Leela, "Differential and Integral Inequalities, Theory and Applications", Academic Press (1969).

How to cite this article:

P. U. Chopade (2018) 'Some Stability Criterion for the Solutions of First Order Difference Equation', *International Journal of Current Advanced Research,* 07(1), pp. 9267-9270. DOI: http://dx.doi.org/10.24327/ijcar.2018.9270.1526

****