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Research Article

ON THE EXTENDIBILITY OF THE SEQUENCES OF DIOPHANTINE TRIPLES INTO QUADRUPLES INVOLVING PELL NUMBERS

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ABSTRACT

In this paper, we search for the sequence of triples involving Pell numbers $\{a,b,c\},\{b,c,d\},\{c,d,e\},\{d,e,f\},........$ such that the product of any two of them added with one is a perfect square. Also, we prove that this sequence of triple can be extended to a sequence of quadruples with the same property.

INTRODUCTION

Let n be an integer. A set of positive integers $(a_1,a_2,a_3,........,a_m)$ is said to have the property D(n) if a_ia_j+n is a perfect square for all $1\le i<j\le m$; such a set is called a Diophantine $m-$ tuple or a p_n set of size m .The problem of construction of such set was studied by Diophantus. He studied the following problem. Find four (positive rational) numbers such that the product of any two of them increased by 1 is a perfect square. He obtained the following solution: $\frac{1}{2},\frac{33}{16},\frac{17}{4},\frac{105}{16}$ (see [1]). The first set of four positive integers with the above property was found by Fermat, and it was $\{1,3,8,120\}$. Euler gave the solution $\{a,b,a+b+2r,4r(r+a)(r+b)\}$, where $ab+1=r^2$ $(see\ [2])$. For an extensive review of various articles one may refer [3- 18]. In this communication we extend the Diophantine triple involving Pell numbers to a quadruple with the property $D(1)$.

Method of Analysis

Let $a=p_{2n}, b=p_{2n+2}$ where $p_n=\frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{2\sqrt{2}}$ be any two integers such that $ab+1$ is a perfect square.

Let c be the non –zero integer such that

$$cp_{2n}+1=\alpha^2 \tag{1}$$

$$cp_{2n+2}+1=\beta^2 \tag{2}$$

From (1), we get

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$$c = \frac{\alpha^2 - 1}{p_{2n}} \tag{3}$$

Substituting (3) in (2), we notice that

$$\left(\alpha^2 - 1\right) p_{2n+2} + p_{2n} = p_{2n}\beta^2 \tag{4}$$

Let

$$\alpha = X + p_{2n}T \tag{5}$$

$$\beta = X + p_{2n+2}T \tag{6}$$

In view of (5) and (6) in (4), we obtain

$$X^2 = DT^2 + 1 \text{ where } D = ab \tag{7}$$

Choosing the initial solution to the Pellian equation (7), as

$X_0 = p_{2n+1}$, $T_0 = 1$,

we get

$$\alpha = p_{2n+1} + p_{2n}$$

$$\beta = p_{2n+1} + p_{2n+2}$$

Substituting the values of α in (1), we get

$$c = p_{2n} + 2p_{2n+1} + p_{2n+2}$$

Thus, we notice that $\{p_{2n}, p_{2n+2}, p_{2n}+2p_{2n+1}+p_{2n+2}\}$ is a Diophantine triple with the property D(1).

Let d be the non –zero integer such that

$$db + 1 = \alpha_1^2 \tag{8}$$

$$dc + 1 = \beta_1^2 \tag{9}$$

From (8), we get

$$d = \frac{\alpha_1^2 - 1}{b} \tag{10}$$

Now, Let e be the non –zero integer such that

$$ec + 1 = \alpha_2^2 \tag{11}$$

$$ed + 1 = \beta_2^2 \tag{12}$$

From (11), we get

$$e = \frac{\alpha_2^2 - 1}{c} \tag{13}$$

Let f be the non –zero integer such that

$$fd + 1 = \alpha_3^2 \tag{14}$$

$$fe + 1 = \beta_3^2 \tag{15}$$

From (14), we get

$$f = \frac{\alpha_3^2 - 1}{d} \tag{16}$$

Choose

$$\alpha_1 = p_{2n+1} + 2p_{2n+2}\ , \beta_1 = p_{2n} + 3p_{2n+1} + 2p_{2n+2}$$

$$\alpha_2 = p_{2n+2} + p_{2n+3} + c\ ,\ \beta_2 = p_{2n+2} + p_{2n+3} + d$$

$$\alpha_3 = p_{2n+2} + p_{2n+3} + p_{2n+4} + d\ , \beta_3 = p_{2n+2} + p_{2n+3} + p_{2n+4} + e$$

and applying the same procedure as explained above, we evaluate that

$$d = p_{2n} + 4p_{2n+1} + 4p_{2n+2}$$

$$e = 2p_{2n} + 6p_{2n+1} + 7p_{2n+2} + 2p_{2n+3}$$

$$f = 3p_{2n} + 10p_{2n+1} + 13p_{2n+2} + 4p_{2n+3} + 2p_{2n+4}$$

Thus, we get the following sequence of triples with the property $D(1)$.

$\{p_{2n+2},\ p_{2n}+2p_{2n+1}+p_{2n+2},\ p_{2n}+4p_{2n+1}+4p_{2n+2}\}$, $\{p_{2n}+2p_{2n+1}+p_{2n+2},\ p_{2n}+4p_{2n+1}+4p_{2n+2},\ 2p_{2n}+6p_{2n+1}+7p_{2n+2}+2p_{2n+3}\}$, $\{p_{2n}+4p_{2n+1}+4p_{2n+2},\ 2p_{2n}+6p_{2n+1}+7p_{2n+2}+2p_{2n+3},\ 3p_{2n}+10p_{2n+1}+13p_{2n+2}+4p_{2n+3}+2p_{2n+4}\}$

Hence, we acquire that

$\{a, b, c\}, \{b, c, d\}, \{c, d, e\}, \{d, e, f\}$,........ is a sequence of triples with the property $D(1)$.

The sequence of triples can be extended to a sequence of quadruples as follows

Let u be any non-zero integer such that

$$ua + 1 = \delta_1^2 \tag{17}$$

$$ub + 1 = \delta_2^2 \tag{18}$$

$$uc + 1 = \delta_3^2 \tag{19}$$

By using the initial solution to (7), its general solution is represented by

$$X_n = \frac{1}{2}\left[\left(p_{2n+1} + \sqrt{D}\right)^{n+1} + \left(p_{2n+1} - \sqrt{D}\right)^{n+1}\right]$$

$$T_n = \frac{1}{2\sqrt{D}}\left[\left(p_{2n+1} + \sqrt{D}\right)^{n+1} - \left(p_{2n+1} - \sqrt{D}\right)^{n+1}\right]$$

Therefore,

$$X_1 = p_{2n+1}^2 + D$$

$$T_1 = 2p_{2n+1}$$

Take

$$\delta_1 = X_1 + aT_1 = p_{2n+1}^2 + D + 2ap_{2n+1} \tag{20}$$

$$\delta_2 = X_1 + bT_1 = p_{2n+1}^2 + D + 2bp_{2n+11} \tag{21}$$

$$\delta_3 = X_1 + cT_1 = p_{2n+1}^2 + D + 2cp_{2n+1} \tag{22}$$

Substituting (20) in (17), we obtain

$$u = p_{2n+2}\left[2p_{2n+1}^2 + 2 + 4p_{2n}p_{2n+1} + 2p_{2n+2}p_{2n}\right] + 4p_{2n}p_{2n+1}^2 + 4p_{2n+1}^3$$

Thus, we get

$\{p_{2n},\ p_{2n+2},\ p_{2n}+2p_{2n+1}+p_{2n+2},\ p_{2n+2}\left[2p_{2n+1}^{2}+2+4p_{2n}p_{2n+1}+2p_{2n+2}p_{2n}\right]+4p_{2n}p_{2n+1}^{2}+4p_{2n+1}^{3}\}$ is a Diophantine quadruple with the property $D(1)$.

Let v be any non-zero integer such that

$$vb+1=\eta_1^2$$
$$vc+1=\eta_2^2$$
$$vd+1=\eta_3^2$$

Let w be any non-zero integer such that

$$wc+1=\varphi_1^2$$
$$wd+1=\varphi_2^2$$
$$we+1=\varphi_3^2$$

Let x be any non-zero integer such that

$$xd+1=\mu_1^2$$
$$xe+1=\mu_2^2$$
$$xf+1=\mu_3^2$$

Consider

$$\eta_1=\left(p_{2n+1}+p_{2n+2}\right)^2+D+2b\left(p_{2n+1}+p_{2n+2}\right)\qquad \eta_2=\left(p_{2n+1}+p_{2n+2}\right)^2+D+2c\left(p_{2n+1}+p_{2n+2}\right)$$
$$\eta_3=\left(p_{2n+1}+p_{2n+2}\right)^2+D+2d\left(p_{2n+1}+p_{2n+2}\right)$$
$$\varphi_1=\left(p_{2n+2}+p_{2n+3}\right)^2+D+2c\left(p_{2n+2}+p_{2n+3}\right)$$
$$\varphi_2=\left(p_{2n+2}+p_{2n+3}\right)^2+D+2d\left(p_{2n+2}+p_{2n+3}\right)$$
$$\varphi_3=\left(p_{2n+2}+p_{2n+3}\right)^2+D+2e\left(p_{2n+2}+p_{2n+3}\right)$$
$$\mu_1=\left(p_{2n+2}+p_{2n+3}+p_{2n+4}\right)^2+D+2d\left(p_{2n+2}+p_{2n+3}+p_{2n+4}\right)$$
$$\mu_2=\left(p_{2n+2}+p_{2n+3}+p_{2n+4}\right)^2+D+2e\left(p_{2n+2}+p_{2n+3}+p_{2n+4}\right)$$
$$\mu_3=\left(p_{2n+2}+p_{2n+3}+p_{2n+4}\right)^2+D+2f\left(p_{2n+2}+p_{2n+3}+p_{2n+4}\right)$$

and repeating the same procedure as explained above, we find that

$$v=\Big(4p_{2n+2}\left(p_{2n}+2p_{2n+1}+p_{2n+2}\right)^2+\left[p_{2n+1}+p_{2n+2}\right]\left(4p_{2n+2}\left(p_{2n+1}+p_{2n+2}\right)+8p_{2n+2}\left(p_{2n}+2p_{2n+1}+p_{2n+2}\right)+4\right)$$
$$+4\left(p_{2n}+2p_{2n+1}+p_{2n+2}\right)\Big)$$
$$w=\Big(4\left(p_{2n}+2p_{2n+1}+p_{2n+2}\right)\left(p_{2n}+4p_{2n+1}+4p_{2n+2}\right)^2+$$
$$\left(p_{2n+2}+p_{2n+3}\right)\left[4\left(p_{2n}+2p_{2n+1}+p_{2n+2}\right)\left(p_{2n+2}+p_{2n+3}\right)+8\left(p_{2n}+2p_{2n+1}+p_{2n+2}\right)\left(p_{2n}+4p_{2n+1}+4p_{2n+2}\right)+4\right]+$$
$$4\left(p_{2n}+4p_{2n+1}+4p_{2n+2}\right)\Big)$$
$$x=\Big(4\left(p_{2n}+4p_{2n+1}+4p_{2n+2}\right)\left(2p_{2n}+6p_{2n+1}+7p_{2n+2}+2p_{2n+3}\right)^2+$$
$$\left(p_{2n+2}+p_{2n+3}+p_{2n+4}\right)\begin{bmatrix}4\left(p_{2n}+4p_{2n+1}+4p_{2n+2}\right)\left(p_{2n+2}+p_{2n+3}+p_{2n+4}\right)+\\ 8\left(p_{2n}+4p_{2n+1}+4p_{2n+2}\right)\left(2p_{2n}+6p_{2n+1}+7p_{2n+2}+2p_{2n+3}\right)+4\end{bmatrix}+$$
$$4\left(2p_{2n}+6p_{2n+1}+7p_{2n+2}+2p_{2n+3}\right)\Big)$$

Thus, we obtain the following sequence of quadruples with the property $D(1)$.

$$\Big(p_{2n+2},\ p_{2n}+2p_{2n+1}+p_{2n+2},\ p_{2n}+4p_{2n+1}+4p_{2n+2},\ 4p_{2n+2}\left(p_{2n}+2p_{2n+1}+p_{2n+2}\right)^2+$$
$$\left[p_{2n+1}+p_{2n+2}\right]\left(4p_{2n+2}\left(p_{2n+1}+p_{2n+2}\right)+8p_{2n+2}\left(p_{2n}+2p_{2n+1}+p_{2n+2}\right)+4\right)+4\left(p_{2n}+2p_{2n+1}+p_{2n+2}\right)\Big)$$

$$\big(p_{2n}+2p_{2n+1}+p_{2n+2}\,,\; p_{2n}+4p_{2n+1}+4p_{2n+2}\,,\; 2p_{2n}+6p_{2n+1}+7p_{2n+2}+2p_{2n+3}\,,$$
$$4(p_{2n}+2p_{2n+1}+p_{2n+2})(p_{2n}+4p_{2n+1}+4p_{2n+2})^2+$$
$$(p_{2n+2}+p_{2n+3})[4(p_{2n}+2p_{2n+1}+p_{2n+2})(p_{2n+2}+p_{2n+3})+8(p_{2n}+2p_{2n+1}+p_{2n+2})(p_{2n}+4p_{2n+1}+4p_{2n+2})+4]$$
$$+4(p_{2n}+4p_{2n+1}+4p_{2n+2})\big)$$

$$\big(p_{2n}+4p_{2n+1}+4p_{2n+2}\,,\; 2p_{2n}+6p_{2n+1}+7p_{2n+2}+2p_{2n+3}\,,\; 3p_{2n}+10p_{2n+1}+13p_{2n+2}+4p_{2n+3}+2p_{2n+4}\,,$$
$$4(p_{2n}+4p_{2n+1}+4p_{2n+2})(2p_{2n}+6p_{2n+1}+7p_{2n+2}+2p_{2n+3})^2+$$
$$(p_{2n+2}+p_{2n+3}+p_{2n+4})\begin{bmatrix}4(p_{2n}+4p_{2n+1}+4p_{2n+2})(p_{2n+2}+p_{2n+3}+p_{2n+4})+\\ 8(p_{2n}+4p_{2n+1}+4p_{2n+2})(2p_{2n}+6p_{2n+1}+7p_{2n+2}+2p_{2n+3})+4\end{bmatrix}+$$
$$4(2p_{2n}+6p_{2n+1}+7p_{2n+2}+2p_{2n+3})\big)$$

Hence, we attain that

$\{a,b,c,u\},\{b,c,d,v\},\{c,d,e,w\},\{d,e,f,x\}$........ is a sequence of quadruples such that the product of any two of them increased by 1 is a perfect square.

Some numerical examples for the above sequences of Diophantine quadruple with property $D(1)$ are presented below

n	(a,b,c,u)	(b,c,d,v)	(c,d,e,w)	(d,e,f,x)
1	$(2,12,24,2380)$	$(12,24,70,80852)$	$(24,70,176,183260)$	$(70,176,468,\ 23064468)$
2	$(12,70,140,470844)$	$(70,140,408,15994836)$	$(140,408,1026,234423628)$	$(408,1026,2728,\ 4567858820)$
3	$(70,408,816,932224$	$(408,816,2378,3166818$	$(816,2378,5980,4\cdot 641553451$	$(2378,5980,15900,\ 9\cdot 044200325\times 10^{11})$

CONCLUSION

In this communication, we have exhibited the sequence of quadruple involving Pell numbers with the property $D(1)$. To conclude, one may search for the sequence of quadruples and quintuples consist of various numbers with some other properties.

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