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Research Article

SCHUR-CONVEXITY FOR GINI MEAN OF n VARIABLES

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ABSTRACT

Schur-convexity, Schur-geometric convexity and Schur-harmonic convexity for Gini mean of *n* variables are investigated, and some mean value inequalities of n variables are established.

1. INTRODUCTION

Throughout the paper we denote the set of n-dimensional row vector on the real number field by R^n.
Also,

$$R_+^n = \left\{ X = (x_1.......x_n) \in R^n : \ x_i > 0 \ i = 1,2,3,...,n \right\}.$$

Let $p,q \in R$ and $a,b \in R_+ := (0,\infty)$ The Gini Means[47] are defined as

$$G_{p,q}(a,b) = \begin{cases} \left(\dfrac{x^p + y^p}{x^q + y^q} \right)^{1/(p-q)}, & p \neq q \\ \exp\left(\dfrac{x^p \ln x + y^p \ln y}{x^q + y^q} \right), & p = q \end{cases} \tag{1.1}$$

It is easy to see that the Gini means $G_{p,q}(a,b)$ are continuous on the domain $\{(a,b;p,q): a,b \in R_+; p,q \in R\}$ and differentiable with respect to $(a,b) \in R^2{}_+$ for fixed $p,q \in R$. Also,Gini means are symmetric with respect to *a*,*b* and *p*,*q*.

Gini means $G_{p,q}(a,b)$ contain many classical two variable means, for example

$$G_{1,0}(x,y) = \frac{x+y}{2} = A(x,y) \text{ is the arithmetic mean,}$$
$$G_{0,0}(x,y) = \sqrt{xy} = G(x,y) \text{ is the geometric mean,}$$

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$$G_{-1,0}(x,y)=\frac{2xy}{x+y}=H(x,y)$$ is the harmonic mean

and more generally, the p-th power mean is equal to $G_{p,p-1}(x,y)=\frac{x^p+y^p}{x^{p-1}+y^{p-1}}$ is the Lehmer mean.

The basic properties of Gini means, as well as their comparison theorems, log-convexities, and inequalities are studied in papers [8, 9, 10, 11, 20, 21, 25, 26, 27, 30, 36, 43, 44, 45, 48].

In recent years Schur-convexity and Schur-geometric convexity of Gini mean have attracted the attention of a considerable number of mathematicians [5, 19, 26, 28, 31, 33]. Sandor proved that the Gini means $G_{p,q}(a,b)$ are Schur convex on $(-\infty,0]\times(-\infty,0]$ and Schur concave on $[-\infty,0)\times[-\infty,0)$ with respect (p,q) for fixed $a,b>0$ with $a\neq b$. Yang improved Sandor's result and proved that Gini means $G_{p,q}(a,b)$ are Schur convex with respect to (p,q) for fixed $a,b>0$ with $a\neq b$ if and only if $p+q<0$ and Schur concave if and only if $p+q>0$. Wang and Zhang [49, 50] showed that Gini means $G_{p,q}(a,b)$ are Schur convex with respect to $(a,b)\in R^2_+$ if and only if $p+q\geq 1$, $p,q\geq 0$ and Schur concave if and only if $p+q\leq 1$, $p\leq 0$ or $p+q\leq 1$, $q\leq 0$.Gu and Shi [12,25] also discussed the Schur convexity. Recently Chu and Xia [6] also proved the same results as Wang and Zhang's.

The Schur geometrically convexity was introduced by Zhang [50]. Wang and Zhang [49] proved Gini means $G_{p,q}(a,b)$ are Schur geometrically convex with respect to $(a,b)\in R^2_+$ if $p+q\geq 0$ and Schur geometrically concave if $p+q\leq 0$.Gu and Shi [12,25] also investigated Schur geometrically convexities of Lehmer mean $G_{p,1-p}(a,b)$ and Gini mean $G_{p,q}(a,b)$ respectively.

Investigation of the elementary properties and inequalities for $L_p(x;y)$ has attracted the attention of a considerable number of mathematicians (see [1, 3, 10, 12, 14, 21, 23, 26, 28, 31]).

In 2009, Gu and Shi [11] discussed the Schur convexity and Schur geometric convexity of the Lehmer means $L_p(x,y)$ with respect to $(x,y)\in R^2_+$ for fixed p. Subsequently, Xia and Chu [36] researched the Schur harmonic convexity of the Lehmer means $L_p(x,y)$ with respect to $(x,y)\in R^2_+$ for fixed p.

In 2016, Chun-Ru Fu and *et al*[51], defined Lehmer mean of n variables $L_p(x)$ on certain subsets of R^n_+ as follows

$$L_p(x)=L_p(x_1,x_2,...,x_n)=\frac{\sum_{i=1}^{n}x_i^p}{\sum_{i=1}^{n}x_i^{p-1}} \qquad (1.2)$$

and studied Schur-convexity, Schur-geometric convexity and Schur-harmonic convexity for Lehmer mean of n variables $L_p(x)$ on certain subsets of R^n_+, and also established some interesting inequalities. This paper motivated us to study about Schur-convexity for Gini mean of n variables.

Let $x=(x_1.......x_n)\in R_+{}^n$. For Schur-convexity and Schur-geometric convexity of n variables Gini mean, and consider $p=1+q$, then

$$G_q(x)=G_q(x_1,x_2,...,x_n)=\frac{\sum_{i=1}^{n}x_i^{q+1}}{\sum_{i=1}^{n}x_i^{q}} \qquad (1.3)$$

K .M Nagaraja and P Siva Kota Reddy [46] obtained the following results.

Lemma 1.1[46]: For $a,b>0$, then the sequence $g_n=\sum_{n=0}^{\infty}(a^n+b^n)$ is log convex.

Lemma 1.2 [46]: For $a,b>0$, then the generalized Contra-harmonic mean $C_n(a,b)=\frac{a^n+b^n}{a^{n-1}+b^{n-1}}$

is increasing with respect to the parameter n, that is $C_{n+1}(a,b)>C_n(a,b)$ for all real n.

Theorem 1.3: The generalized Contra-harmonic mean is monotonically increasing with respect to the parameter n if and only if the sequence g_n of Lemma 1.1 is log-convex.

Remark: $L_p(x) \le G_q(x)$

Proof: Let $g_n = \left(a^n + b^n\right)$, consider

$$g_n^{\;2} - g_{n+1}\, g_{n-1} = \left(a^n + b^n\right)^2 - \left(a^{n+1} + b^{n+1}\right)\left(a^{n-1} + b^{n-1}\right)$$
$$= a^{n-1}b^{n-1}\left[2ab - a^2 - b^2\right]$$
$$= -a^{n-1}b^{n-1}\left(a-b\right)^2 \le 0.$$

This proves that $g_n^{\;2} \le g_{n+1}\, g_{n-1}$. Substitute $g_n = a^n + b^n$.

Then, $\dfrac{a^n + b^n}{a^{n-1} + b^{n-1}} \le \dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}$.

This implies that, $\dfrac{\sum_{i=1}^{n} x_i^{p}}{\sum_{i=1}^{n} x_i^{p-1}} \le \dfrac{\sum_{i=1}^{n} x_i^{q+1}}{\sum_{i=1}^{n} x_i^{q}}$

i.e., $L_p(x) \le G_q(x)$.

In this paper, we study Schur-convexity, Schur-geometric convexity and Schur-harmonic convexity of $G_q(x)$ on certain subsets of R_+^n. As consequences, some interesting inequalities are obtained.

2. DEFINATION AND LEMMA

We need the following definitions and lemmas.

Definition 2.1: **([17,27]).** Let $x = \left(x_1, x_2, x_3, ..., x_n\right)$ and $y = \left(y_1, y_2, y_3, ..., y_n\right) \in R^n$

1. x is said to be majorized by y (in symbols $x \prec y$), $\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}$ for $k = 1, 2, 3..., n-1$ and $\sum_{i=1}^{n} x_i \le \sum_{i=1}^{n} y_i$ where $x_{[1]} \ge ... \ge x_{[n]}$ and $y_{[1]} \ge ... \ge y_{[n]}$ are rearrangement of x and y in a descending order.
2. $\Omega \subset R^n$ is called a convex set, if $\left(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, ..., \alpha x_n + \beta y_n\right) \in \Omega$, for any $\boldsymbol{x}$ and $y \in \Omega$, where α and $\beta \in \left[0,1\right]$ with $\alpha + \beta = 1$
3. Let $\Omega \subset R^n$, the function $\varphi : \Omega \to R^n$ is said to be schur convex function on Ω if $x \prec y$ on Ω implies $\varphi(x) \le \varphi(y)$. φ is said to be a Schur concave function on Ω, if and only if $-\varphi$ is Schur convex function.

Definition 2.2: **([20,44]).** Let $x = \left(x_1, x_2, x_3, ..., x_n\right)$ and $y = \left(y_1, y_2, y_3, ..., y_n\right) \in R^n_{\;+}$.

1. $\Omega \subset R^n$ is called geometrically convex set, if $\left(x_1^{\;\alpha} y_1^{\;\beta}, x_2^{\;\alpha} y_2^{\;\beta}, ..., x_n^{\;\alpha} y_n^{\;\beta}\right) \in \Omega$, for any $\boldsymbol{x}$ and $y \in \Omega$, where $\alpha, \beta \in \left[0,1\right]$ with $\alpha + \beta = 1$.
2. Let $\Omega \subset R^n_{\;+}$, the function $\varphi : \Omega \to R_+^{\;n}$ is said to be schur geometrically convex function on Ω if $\left(\ln x_1, \ln x_2, ..., \ln x_n\right) \prec \left(\ln y_1, \ln y_2, ..., \ln y_n\right)$ on Ω implies $\varphi(x) \le \varphi(y)$. φ is said to be a Schur geometrically concave function on Ω if and only if $-\varphi$ is Schur geometrically convex function.

Definition 2.3: **([4,18]).** Let $x = \left(x_1, x_2, x_3, ..., x_n\right)$ and $y = \left(y_1, y_2, y_3, ..., y_n\right) \in R^n_{\;+}$.

1. A set $\Omega \subset R^n$ is said to be a harmonically convex set, if

$$\left(\frac{x_1 y_1}{\lambda x_1 + (1-\lambda) y_1}, \frac{x_2 y_2}{\lambda x_2 + (1-\lambda) y_2}, ..., \frac{x_n y_n}{\lambda x_n + (1-\lambda) y_n}\right) \in \Omega$$

for any $\boldsymbol{x}$ and $y \in \Omega$, and $\lambda \in$ [0, 1].

2. A function $\varphi:\Omega\to R_+$ is said to be a Schur -harmonically convex function on Ω, if $\left(\frac{1}{x_1},\frac{1}{x_2},...,\frac{1}{x_n}\right)\prec\left(\frac{1}{y_1},\frac{1}{y_2},...,\frac{1}{y_n}\right)$, implies $\varphi(x)\le\varphi(y)$. φ is said to be a Schur harmonically concave function on Ω if and only if $-\varphi$ is a Schur -harmonically convex function.

Lemma 2.4: **([17,27]).** Let $\Omega\subset R^n$ be symmetric with non emptyinterior convex set and let $\varphi:\Omega\to R_+$ be continuous on Ω and differentiable on Ω^0. Then φ is Schur convex (concave) if

$$(x_1-x_2)\left(\frac{\partial\varphi(X)}{\partial x_1}-\frac{\partial\varphi(X)}{\partial x_2}\right)\ge 0(\le 0).$$

holds for any $x=(x_1,x_2,x_3,...,x_n)\in\Omega^0$.

Remark 2.5: **[9,19].** It is easy to see that the condition (2.1) is equivalent to

$$\frac{\partial\phi(x)}{\partial x_i}\le\frac{\partial\phi(x)}{\partial x_{i+1}}\quad(\text{or}\ge\text{ resp. }),\ i=1,...,n-1,\ \text{for all } x\in D\cap\Omega,$$

where $D=\{x:x_1\le x_2\le...\le x_n\}$

The condition (2.1) is also equivalent to

$$\frac{\partial\varphi(X)}{\partial x_{i1}}\ge\frac{\partial\varphi(X)}{\partial x_{i+1}}\quad(\text{or}\ge\text{ resp. }),\ i=1,...,n-1,\ \text{for all } x\in E\cap\Omega,$$

where $E=\{x:x_1\ge x_2\ge...\ge x_n\}$.

Lemma 2.6: **([20,44]).** Let $\Omega\subset R^n$ be a symmetric geometrically convex set with non empty interior Ω^0. Let $\varphi:\Omega\to R_+$ be continuous on Ω and differentiable on Ω^0. Then φ is Schurgemetrically convex (concave) function $x=(x_1,x_2,x_3,...,x_n)\in\Omega^0$ if and only if φ is symmetric on Ω and

$$(x_1-x_2)\left(x_1\frac{\partial\varphi(X)}{\partial x_1}-x_2\frac{\partial\varphi(X)}{\partial x_2}\right)\ge 0(\le 0).$$

holds for any $x=(x_1,x_2,x_3,...,x_n)\in\Omega^0$

Remark 2.7: It is easy to see that the condition (2.2) is equivalent to

$$x_i\frac{\partial\phi(x)}{\partial x_i}\le x_{i+1}\frac{\partial\phi(x)}{\partial x_{i+1}}\quad(\text{or}\ge\text{ resp. }),\ i=1,...,n-1,\ \text{for all } x\in D\cap\Omega,$$

where $D=\{x:x_1\le x_2\le...\le x_n\}$

The condition (2.2) is also equivalent to

$$x_i\frac{\partial\phi(x)}{\partial x_i}\ge x_{i+1}\frac{\partial\phi(x)}{\partial x_{i+1}}\quad(\text{or}\ge\text{ resp. }),\ i=1,...,n-1,\ \text{for all } x\in E\cap\Omega,$$

where $E=\{x:x_1\ge x_2\ge...\ge x_n\}$.

Lemma 2.8: **([4,18]).** Let $\Omega\subset R^n$ be symmetric harmonically convex set with non empty interior Ω^0. Let $\varphi:\Omega\to R_+$ be continuous on Ω and differentiable on Ω^0. Then φ is Schur harmonically convex (concave) function $x=(x_1,x_2,x_3,...,x_n)\in\Omega^0$ if and only if φ is symmetric on Ω and

$$(x_1-x_2)\left(x_1^2\frac{\partial\varphi(X)}{\partial x_1}-x_2^2\frac{\partial\varphi(X)}{\partial x_2}\right)\ge 0(\le 0).$$

holds for any $x=(x_1,x_2,x_3,...,x_n)\in\Omega^0$

Remark 2.9: It is easy to see that the condition (2.3) is equivalent to

$$x_i^2\frac{\partial\phi(x)}{\partial x_i}\le x_{i+1}^2\frac{\partial\phi(x)}{\partial x_{i+1}}\quad (\text{or}\ge\text{ resp. }),\ i=1,...,n\text{-}1,\ \text{for all } x\in D\cap\Omega,$$

where $D=\{x:x_1\le x_2\le...\le x_n\}$

The condition (2.3) is also equivalent to

$$x_i^2\frac{\partial\phi(x)}{\partial x_i}\ge x_{i+1}^2\frac{\partial\phi(x)}{\partial x_{i+1}}\quad (\text{or}\ge\text{ resp. }),\ i=1,...,n\text{-}1,\ \text{for all } x\in E\cap\Omega,$$

where $E=\{x:x_1\ge x_2\ge...\ge x_n\}$.

Lemma 2.10: Let $x_1\ge x_2\ge...\ge x_n>0,\ m\in R.$ Then

$$x_1\ge\frac{x_1^m+x_2^m+...+x_n^m}{x_1^{m-1}+x_2^{m-1}+...+x_n^{m-1}}\ge x_n.$$

Proof

$$x_1\left(x_1^{m-1}+x_2^{m-1}+...+x_n^{m-1}\right)-\left(x_1^m+x_2^m+...+x_n^m\right)$$
$$=x_1^{m-1}(x_1-x_1)+x_2^{m-1}(x_1-x_2)+...+x_n^{m-1}(x_1-x_n)\ge 0,$$
$$x_n\left(x_1^{m-1}+x_2^{m-1}+...+x_n^{m-1}\right)-\left(x_1^m+x_2^m+...+x_n^m\right)$$
$$=x_1^{m-1}(x_n-x_1)+x_2^{m-1}(x_n-x_2)+...+x_n^{m-1}(x_n-x_n)\le 0.$$

This is proof of Lemma 2.10

Lemma 2.11: **([17]).** Let $x=(x_1,x_2,...,x_n)\in R_+^n$ and $A_n(x)=\frac{1}{n}\sum_{i=1}^n x_i$. Then

$$u=\underbrace{(A_n(x),A_n(x),...,A_n(x),)}_{n}<(x_1,x_2,...,x_n)=x.$$

3. MAIN RESULTS

Theorem 3.1 Let $x=(x_1,x_2,...,x_n)\in R_+^{\ n},\ n\ge 2$ and $q\in R$.

(I) If $q\ge 1$, then for any $a>0, G_q(x)$ is Schur-convex with $x\in\left(\frac{(q\text{-}1)a}{q+1},a\right)^n$

(II) If $q<0$, then for any $a>0, G_q(x)$ is Schur-concave with $x\in\left(a,\frac{(q-1)a}{q+1}\right)^n$

Proof

Straightforward computation gives

$$\frac{\partial G_q(x)}{\partial x_i}=\frac{(q+1)x_i^q\sum_{j=1}^n x_j^q-qx_i^{q-1}\sum_{j=1}^n x_j^{q+1}}{\left(\sum_{j=1}^n x_j^q\right)^2}\qquad i=1,2,...,n, \tag{3.1.1}$$

and then

$$\frac{\partial G_q(x)}{\partial x_i}-\frac{\partial G_q(x)}{\partial x_{i+1}}=\frac{f_i(x)}{\left(\sum_{j=1}^n x_j^q\right)^2}\qquad i=1,2,...,n,$$

where $f_i(x) = (q+1)x_i^{q}\sum_{j=1}^{n} x_j^{q} - qx_i^{q-1}\sum_{j=1}^{n} x_j^{q+1}$.

It is clear that $G_q(x)$ is symmetric with $x \in R_+^n$. Without loss of generality, we may assume that $x \geq x_2 \geq \ldots \geq x_n > 0.$

For any $a > 0$, according to the integral mean value theorem, there is a ξ which lies between x_i and x_{i+1} such that

$$(q+1)\left(x_i^{q} - x_{i+1}^{q}\right) - aq\left(x_i^{q-1} - x_{i+1}^{q-1}\right) = q(q+1)\int_{x_{i+1}}^{x_i} x^{q-1}dx - a(q-1)(q)\int_{x_{i+1}}^{x_i} x^{q-2}dx$$

$$= q\int_{x_{i+1}}^{x_i}\left[(q+1)x^{q+1} - a(q-1)\ x^{q-2}\right]dx \qquad (3.1.2)$$

$$= q\left[(q+1)\,\xi^{q-1} - a\,(q-1)\,\xi^{q-2}\right](x_i - x_{i+1})$$

$$= q\ (q+1)\xi^{q-2}\left(\xi - \frac{(q-1)a}{q+1}\right)(x_i - x_{i+1})$$

Proof of (I): When $q \geq 1$ and $a \geq x_1 \geq x_2 \geq \ldots \geq x_n \geq \frac{(q-1)a}{q+1} > 0$, from 3.1.2 we have

$$(q+1)\left(x_i^{q} - x_{i+1}^{q}\right) - aq\left(x_i^{q-1} - x_{i+1}^{q-1}\right) \geq 0$$

that is

$$\frac{(q+1)\left(x_i^{q} - x_{i+1}^{q}\right)}{q\left(x_i^{q-1} - x_{i+1}^{q-1}\right)} \geq a$$

and then from Lemma 2.10 it follows that

$$\frac{(q+1)\left(x_i^{q} - x_{i+1}^{q}\right)}{q\left(x_i^{q-1} - x_{i+1}^{q-1}\right)} \geq x_1 \geq \frac{\sum_{j=1}^{n} x_j^{q+1}}{\sum_{j=1}^{n} x_j^{q}},$$

namely, $f_i(x) \geq 0$, and then $\frac{\partial G_q(x)}{\partial x_i} \geq \frac{\partial G_q(x)}{\partial x_{i+1}}$.

By Lemma 2.4 it follows that $G_q(x)$ is Schur-convex with $x \in \left[\frac{(q-1)a}{q+1}, a\right]^n$.

***Proof of (II)**:* When $q < 0$ and $\frac{(q-1)a}{q+1} \geq x_1 \geq x_2 \geq \ldots \geq x_n \geq a > 0$,

$$(q+1)\left(x_i^{q} - x_{i+1}^{q}\right) - a\ q\left(x_i^{q-1} - x_{i+1}^{q-1}\right) \leq 0$$

and then from Lemma 2.5, it follows that

$$\frac{(q+1)\left(x_i^{q} - x_{i+1}^{q}\right)}{q\left(x_i^{q-1} - x_{i+1}^{q-1}\right)} \leq x_n \leq \frac{\sum_{j=1}^{n} x_j^{q+1}}{\sum_{j=1}^{n} x_j^{q}},$$

namely,

$f_i(x) \geq 0$, and then $\frac{\partial G_q(x)}{\partial x_i} \leq \frac{\partial G_q(x)}{\partial x_{i+1}}$.

By Lemma 2.4 it follows that $G_q(x)$ is Schur-concave with $x \in \left[a, \frac{(q-1)a}{q+1}\right]^n$.

The proof of Theorem 3.1 is complete.

Theorem 3.2 Let $x = (x_1, x_2, \ldots, x_n) \in R_+^n$, $n \geq 2$ and $q \in R$.

(I) If $q < \dfrac{1}{2}$ and $q \neq 0,$ then for any $a > 0, G_q(x)$ is Schur-geometrically concave with $x \in \left(a, a\left(\dfrac{q}{q+1}\right)^2 \right)^n$

(II) If $q > \dfrac{1}{2}$, then for any $a > 0, G_q(x)$ is Schur-geometrically convex with $x \in \left(a\left(\dfrac{q}{q+1}\right)^2, a \right)^n.$

Proof

From (3.1.1), we have

$$x_i \frac{\partial G_q(x)}{\partial x_i} - x_{i+1}\frac{\partial G_q(x)}{\partial x_{i+1}} = \frac{g_i(x)}{\left(\sum_{j=1}^{n} x_j^{\,q}\right)^2} \qquad i = 1, 2, \ldots, n,$$

where

$$g_i(x) = (q+1)\left(x_i^{\,q+1} - x_{i+1}^{\,q+1}\right)\sum_{j=1}^{n} x_j^{\,q} - q\left(x_i^{\,q} - x_{i+1}^{\,q}\right)\sum_{j=1}^{n} x_j^{\,q+1}.$$

It is clear that $G_q(x)$ is symmetric with $x \in R_+^n$. Without loss of generality, we may assume that $x \geq x_2 \geq \ldots \geq x_n > 0.$

For any $a > 0$, according to the integral mean value theorem, there is a ξ which lies between x_i and x_{i+1} such that

$$(q+1)\left(x_i^{\,q+1} - x_{i+1}^{\,q+1}\right) - aq\left(x_i^{\,q} - x_{i+1}^{\,q}\right) = (q+1)^2 \int_{x_{i+1}}^{x_i} x^q dx - a\left(q^2\right)\int_{x_{i+1}}^{x_i} x^{q-1} dx$$

$$= \int_{x_{i+1}}^{x_i}\left[(q+1)^2 x^q - a\left(q^2\right) x^{q-1}\right]dx \tag{3.2.1}$$

$$= \left[(q+1)^2 \xi^q - a\left(q^2\right)\xi^{q-1}\right]\left(x_i - x_{i+1}\right)$$

$$= (q+1)^2 \xi^{q-1}\left(\xi - \left(\frac{q}{q+1}\right)^2 a\right)\left(x_i - x_{i+1}\right)$$

Proof of (I): When $q > \dfrac{1}{2}$ and $a \geq x_1 \geq x_2 \geq \ldots \geq x_n \geq \left(\dfrac{q}{q+1}\right)^2 a > 0$, from 3.2.1 we have

$$(q+1)\left(x_i^{\,q+1} - x_{i+1}^{\,q+1}\right) - aq\left(x_i^{\,q} - x_{i+1}^{\,q}\right) \geq 0$$

that is

$$\frac{(q+1)\left(x_i^{\,q+1} - x_{i+1}^{\,q+1}\right)}{q\left(x_i^{\,q} - x_{i+1}^{\,q}\right)} \geq a$$

and then from Lemma 2.10, it follows that

$$\frac{(q+1)\left(x_i^{\,q+1} - x_{i+1}^{\,q+1}\right)}{q\left(x_i^{\,q} - x_{i+1}^{\,q}\right)} \geq x_1 \geq \frac{\sum_{j=1}^{n} x_j^{\,q+1}}{\sum_{j=1}^{n} x_j^{\,q}},$$

namely, $g_i(x) \geq 0$, and then $x_i \dfrac{\partial G_q(x)}{\partial x_i} \geq x_{i+1}\dfrac{\partial G_q(x)}{\partial x_{i+1}}$. By Lemma 2.6 and remark 2.7 it follows that $G_q(x)$ is Schur-geometrically convex with $x \in \left[\left(\dfrac{q}{q+1}\right)^2 a, a\right]^n$.

Proof of (II): When $q < \frac{1}{2}$ and $\left(\frac{q}{q+1}\right)^2 a \geq x_1 \geq x_2 \geq \ldots \geq x_n \geq a > 0$,

$(q+1)\left(x_i^{q+1} - x_{i+1}^{q+1}\right) - aq\left(x_i^{q} - x_{i+1}^{q}\right) \leq 0$

and then from Lemma 210, it follows that

$$\frac{(q+1)\left(x_i^{q+1} - x_{i+1}^{q+1}\right)}{q\left(x_i^{q} - x_{i+1}^{q}\right)} \leq x_n \leq \frac{\sum_{j=1}^{n} x_j^{q+1}}{\sum_{j=1}^{n} x_j^{q}},$$

namely,

$g_i(x) \geq 0$, and then $x_i \frac{\partial G_q(x)}{\partial x_i} \leq x_{i+1} \frac{\partial G_q(x)}{\partial x_{i+1}}$.

By Lemma 2.6 it follows that $G_q(x)$ is Schur- geometrically concave with $x \in \left[a, \left(\frac{q}{q+1}\right)^2 a\right]^n$.

Proof of (III): When $q=0$ $g_i(x) \leq 0$ it follows that $G_q(x)$ is Schur- geometrically concave with $x \in R_+^n$

The proof of Theorem 3.2 is complete.

Theorem 3.3. Let $x = (x_1, x_2, \ldots, x_n) \in R_+^{\,n}, n \geq 2$ and $q \in R$.

(I) If $q > 2$, then for any $a > 0, G_q(x)$ is Schur-Harmonically convex with $x \in \left(\frac{(q)a}{q+2}, a\right)^n$

(II) If $q < -2$, then for any $a > 0, G_q(x)$ is Schur-Harmonically concave with $x \in \left(a, \frac{(q)a}{q+2}\right)^n$

Proof

From (3.1.1), we have

$$x_i^2 \frac{\partial G_q(x)}{\partial x_i} - x_{i+1}^2 \frac{\partial G_q(x)}{\partial x_{i+1}} = \frac{h_i(x)}{\left(\sum_{j=1}^{n} x_j^q\right)^2} \qquad i = 1, 2, \ldots, n-1,$$

where

$$h_i(x) = (q+1)\left(x_i^{q+2} - x_{i+1}^{q+2}\right)\sum_{j=1}^{n} x_j^q - q\left(x_i^{q+1} - x_{i+1}^{q+1}\right)\sum_{j=1}^{n} x_j^{q+1}.$$

It is clear that $G_q(x)$ is symmetric with $x \in R_+^n$. Without loss of generality, we may assume that

$x \geq x_2 \geq \ldots > x_n > 0.$

For any $a > 0$, according to the integral mean value theorem, there is a ξ which lies between x_i and x_{i+1} such that

$$(q+1)\left(x_i^{q+2} - x_{i+1}^{q+2}\right) - a\,q\left(x_i^{q+1} - x_{i+1}^{q+1}\right) = (q+1)(q+2)\int_{x_{i+1}}^{x_i} x^{q+1}\,dx - a\,q(q+1)\int_{x_{i+1}}^{x_i} x^q dx$$

$$= \int_{x_{i+1}}^{x_i} \left[(q+1)(q+2)x^{q+1} - a\,q(q+1)x^q\right]dx \qquad (3.3.1)$$

$$= \left[(q+1)(q+2)\xi^{q+1} - a\,q\,(q+1)\xi^q\right](x_i - x_{i+1})$$

$$= \xi^q\left(\xi - \left(\frac{q}{q+2}\right)a\right)(x_i - x_{i+1})$$

Proof of (I): When $q > 2$ and $a \geq x_1 \geq x_2 \geq \ldots \geq x_n \geq \left(\frac{q}{q+2}\right)a > 0$, from 3.3.1 we have

$(q+1)\left(x_i^{q+2} - x_{i+1}^{q+2}\right) - aq\left(x_i^{q+1} - x_{i+1}^{q+1}\right) \geq 0$

that is

$$\frac{(q+1)\left(x_i^{\,q+2}-x_{i+1}^{\,q+2}\right)}{q\left(x_i^{\,q+1}-x_{i+1}^{\,q+1}\right)} \geq a$$

and then from Lemma 2.10, it follows that

$$\frac{(q+1)\left(x_i^{\,q+2}-x_{i+1}^{\,q+2}\right)}{q\left(x_i^{\,q+1}-x_{i+1}^{\,q+1}\right)} \geq x_1 \geq \frac{\sum_{j=1}^{n} x_j^{\,q+1}}{\sum_{j=1}^{n} x_j^{\,q}},$$

namely, $h_i(x)\geq 0$, and then $x_i^2 \frac{\partial G_q(x)}{\partial x_i} \geq x_{i+1}^2 \frac{\partial G_q(x)}{\partial x_{i+1}}$.

By Lemma 2.8 and remark 2.9 it follows that $G_q(x)$ is Schur- harmonically convex with $x \in \left[\left(\frac{q}{q+2}\right)a, a\right]^n$.

Proof of (II): When $q<-2$ and $\left(\frac{q}{q+2}\right)a \geq x_1 \geq x_2 \geq \ldots \geq x_n \geq a > 0$,

$$(q+1)\left(x_i^{\,q+2}-x_{i+1}^{\,q+2}\right) - aq\left(x_i^{\,q+1}-x_{i+1}^{\,q+1}\right) \leq 0$$

and then from Lemma 2.10, it follows that

$$\frac{(q+1)\left(x_i^{\,q+2}-x_{i+1}^{\,q+2}\right)}{q\left(x_i^{\,q+1}-x_{i+1}^{\,q+1}\right)} \leq x_n \leq \frac{\sum_{j=1}^{n} x_j^{\,q+1}}{\sum_{j=1}^{n} x_j^{\,q}},$$

namely, $h_i(x)\geq 0$, and then $x_i^2 \frac{\partial G_q(x)}{\partial x_i} \leq x_{i+1}^2 \frac{\partial G_q(x)}{\partial x_{i+1}}$.

By Lemma 2.8 and Remark 2.9 it follows that $G_q(x)$ is Schur- harmonically concave with $x \in \left[a, \left(\frac{q}{q+2}\right)a\right]^n$.

The Proof of Theorem 3.3 is complete.

4. Applications

Theorem 4.1: If $q \geq 1$, then for any $a>0$, $x \in \left(\frac{(q-1)a}{q+1}, a\right)^n$ then we have

$$A_n(x) \geq G_q(x) \tag{4.1}$$

If $q<0$, and $x \in \left(a, \frac{(q-1)a}{q+1}\right)^n$ then the inequality (4.1) is reversed

Proof: If $q \geq 1$, then for any $a>0$, $x \in \left(\frac{(q-1)a}{q+1}, a\right)^n$ then by theorem 2.5 from Lemma 2.11 we have

$$G_q(u) \geq G_q(x),$$

rearranging gives (4.1) If $q<0$, and $x \in \left(a, \frac{(q-1)a}{q+1}\right)^n$ then the inequality (4.1) is reversed

The proof is complete.

Theorem 4.2: If $q > \frac{1}{2}$, then for any $a>0$, $x \in \left(a\left(\frac{q}{q+1}\right)^2, a\right)^n$ then we have

$$G_n(x) \geq G_q(x) \tag{4.2}$$

where $G_n(x)=\sqrt[n]{x_1x_2\dots x_n}$ is geometric mean of x.

If $q<\frac{1}{2}$, $q\neq 0$ and $x\in\left(a,a\left(\frac{q}{q+1}\right)^2\right)^n$ then the inequality (4.2) is reversed

Proof: By Lemma 2.11 we have

$$\left(\underbrace{\log G_n(x),\dots,\log G_n(x)}_{n}\right)<\left(\log x_1,\log x_2,\dots,\log x_n\right),$$

If $q>\frac{1}{2}$ and, $x\in\left(a\left(\frac{q}{q+1}\right)^2,a\right)^n$, by theorem 2.6 it follows

$$G_q\left(\underbrace{G_n(x),\dots,G_n(x)}_{n}\right)\leq G_q\left(x_1,x_2,\dots,x_n\right),$$

rearranging gives (4.2) If $q<\frac{1}{2},q\neq 0$ and $x\in\left(a,a\left(\frac{q}{q+1}\right)^2\right)^n$ then the inequality (4.2) is reversed.

The proof is complete.

Theorem 4.3: If $q>2,$ then for any $a>0, x\in\left(\frac{(q)a}{q+2},a\right)^n$ then we have

$$H_n(x)\leq G_q(x) \tag{4.3}$$

where $H_n(x)=\frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}$ is the harmonic mean of x.

If $q<-2, x\in\left(a,a\left(\frac{q}{q+2}\right)\right)^n$ then the inequality (4.3) is reversed

Proof: By Lemma 2.11 we have

$$\left(\underbrace{\frac{1}{H_n(x)},\dots,\frac{1}{H_n(x)}}_{n}\right)<\left(\frac{1}{x_1},\dots,\frac{1}{x_n}\right),$$

If $q>2$ and, $x\in\left(a\left(\frac{q}{q+2}\right),a\right)^n$, by theorem 2.7 it follows

$$G_q\left(\underbrace{H_n(x),\dots,H_n(x)}_{n}\right)\leq G_q\left(x_1,x_2,\dots,x_n\right),$$

rearranging gives (4.3) If $q<-2,$ and $x\in\left(a,a\left(\frac{q}{q+2}\right)\right)^n$ then the inequality (4.3) is reversed.

The proof is complete.

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