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Research Article

COMMON FIXED POINT THEOREM IN FUZZY 3 - METRIC SPACES

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We prove a common fixed point theorem in fuzzy metric space and present some common fixed point theorems for weakly compatible mappings in fuzzy 3- metric spaces under various conditions.

INTRODUCTION

The concept of fuzzy Set was introduced by Zadeh [20] in 1965. Kramosil and Michalek [8] introduced fuzzy metric space, George and Veermani [4] modified the notion of fuzzy metric spaces with the help of continuous t-norm. Many researchers have obtained common fixed point theorems for mappings satisfying different types of commutative conditions. Consequently in due course of time some metric fixed point results were generalized to fuzzy metric spaces by various authors. We know that that 2-metric space is a real valued function of a point triples on a set X, which abstract properties were suggested by the area function in Euclidean spaces. Now it is natural to expect 3-Metric space, which is suggested by the volume function. Singhi J, Bhardwaj R, Agrawal S and Shrivastava R [19] studies fuzzy 3 - metric space. In the present paper we are proving a common fixed point theorem for fuzzy3-metric spaces for weakly compatible mapping.

Definitions

Definition: A fuzzy set A on X is a function with domain [0,1].→X and values in [0, 1] i.e. $A : X \to [0,1]$.

Definition: A binary operation *: [0,1] x [0,1] x [0,1] x [0,1] → [0,1] is called a continuous t-norm if ([0,1], *) is an abelian topological monoidies with unit 1 such that $a_1 * b_1 * c_1 * d_4 \geq a_2 * b_2 * c_1 * d_4$, whenever $a_1 \geq a_2$, $b1 \geq b_2$, $c_1 \geq c_2$, $d1 \geq d_2$ for all a_1, a_2, b_1, b_2, c_1 , c_2 and d1 , d_2 are in [0,1].

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Definition: The 3-tuple (X,M,*) is called a fuzzy 3- metric space if X is an arbitrary set, * is continuous t-norm and M is a fuzzy set in $X^4 \times [0,\infty)$ satisfying the followings conditions and t1, t2,t3,t4∈for all x,y,z,w,u > 0

1. M(x,y,z,w,0) = 0
2. M(x,y,z,w,t) = 1 for all t>0
3. M(x,y,z,w,t) = M(x,w,z,y,t) = M(y,z,w,x,t) = M(z,w,x,y,t) = ...
4. $M(x,y,z,w,t_1+t_2+t_3+t_4) \geq M(x,y,z,u,t_1) * M(x,y,u,w,t_2) * (4M(x,u,z,w,t_3) * M(u,y,z,w,t_4)$

Definition: Let (X, M, *) be a fuzzy 3 – metric space. A sequence $\{x_n\}$ in fuzzy 3 – metric space X is said to be convergent to a point x ∈ X, $\lim_{n\to\infty} M(x_n, x, a, b, t) = 1$ for all a, b ∈ X and t > 0.

Definition: Let (X, M, *) be a fuzzy 3 – metric space. A sequence $\{x_n\}$ in fuzzy 3 – metric space X is said to be Cauchy sequence if $\lim_{n\to\infty} M\left(x_{n+p}, x_n, a, b, t\right) = 1$ for all a, b ∈ X and t, p > 0.

Definition*:* A fuzzy 3-metric space in which every sequence is convergent is said to be complete.

Definition: A function M is continuous in fuzzy 3 – metric space if and only if whenever for all a ∈ X and t > 0. $X_n \to x$, $y_n \to y$, then $\lim_{n\to\infty} M(x_n, y_n, a, b, t) = M(x, y, a, b\ t)$ for all a, b ∈ X and t > 0.

Definition: Two mappings A and S on fuzzy 3 – metric space X are weakly commuting if and only if $M(ASu, SAu, a, b, t) \geq M(Au, Su, a, t)$, $\forall$ u, a, b $\in$ X and t > 0.

Definition: Self mappings A and B of 3 – metric space (X, M, *) is said to be compatible, if $\lim_{n\to\infty} M(ABx_n, BAx_n, a, b, t) = 1$ for all a, b $\in$ X and t > 0, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some z $\in$ X.

Definition: Two self mappings A and S of a fuzzy metric space (X, M, ∗) are said to be weakly compatible if, they commute at coincidence points. That is, Ax = Sx implies that ASx = SAx for all x in X. It is important to note that compatible mappings in a metric space are weakly compatible but weakly compatible mappings need not be compatible [24].

Definition: Two self mappings A and S of a fuzzy metric space (X, M, ∗) are said to be semi-compatible if $\lim_{n\to\infty} M(ASxn, Sx,a, b, t) = 1$ whenever $\{x_n\}$ is a sequence such that, $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = p$, for some p in X.

Definition: Two self mappings A and S of a fuzzy metric space (X, M, *) are said to be reciprocally continuous if $\lim_{n\to\infty} M(ASxn, Ax,a,b, t) = 1$ and $\lim_{n\to\infty} M(SAxn, Sx,a,b, t) = 1$ whenever $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = p$, for some p in X.

Lemma

Lemma: Let (X,M,*) be a fuzzy metric space. If there exists k $\in$ (0,1) such that

$M(x,y, a,b,kt) \geq M(x,y,a,b,t)$, then x = y.

Lemma: Let A and S be two self mappings on a fuzzy metric space (X, M, *). If the pair (A, S) is reciprocally continuous, then (A, S) is semi compatible if and only if (A, S) is compatible.

Main Result

Theorem: Let (X,M,*) be a complete fuzzy 3-metric space and let A,B,S and T be self mappings of X satisfying

(4.1.1) $AX \subseteq TX$, $BX \subseteq SX$.

(4.1.2) $M(Ax, By,a, b, t) \geq r[\min\{M(Ax, Sx,a,b,t), M(By, Ty,a, b, t), M(Sx, Ty,a, b, t),$

$M(Ax, Ty,a, b, \alpha t), M(Sx, By, (2 - \alpha)t)\}$

where $r : [0,1] \to [0,1]$ is a continuous function such that r(t) > t, for each $0 \leq t \leq 1$ and r(t) = 1 for t =1and for all x, y $\in$X, $\alpha \in (0, 2)$and t>0. If (A, S) or (B, T) is semi-compatible pair of reciprocally continuous mappings with respectively (B,T) or (A,s) as weakly compatible mappings, then A, B, S and T have a unique common fixed point in X.

Proof: Let $x_0 \in X$ be an arbitrary point. Then since $AX \subseteq TX$, $BX \subseteq SX$, there exists $x_1, x_2 \in X$, such that $Ax_0 = Tx_1$ and $Bx_1 = Sx_2$. Inductively, we construct the sequences $\{y_n\}$ and $\{x_n\}$ in X such that $y_{2n} = Ax_{2n} = Tx_{2n+1}$ and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$, for n = 0, 1, 2, ...

Now we put $\alpha = 1\text{-}q$ with $q \in (0,1)$ in (4.1.2), we get
$M(y_{2n}, y_{2n+1},a, b, t) = M(Ax_{2n}, Bx_{2n+1}, a, b, t)$
$\geq r(\min\{M(Sx_{2n},Tx_{2n+1},a,b, t), M(Bx_{2n+1}, Tx_{2n+1},a, b, t), M(Sx_{2n}, Tx_{2n+1},a, b, t),$
$M(Ax_{2n}, Tx_{2n+1}, a, b. (1 - q)t), M(Sx_{2n}, Bx_{2n+1},a, b, (1 + q)t)\})$.

That is, $M(y_{2n}, y_{2n+1},a, b, t) \geq r(\min\{M(y_{2n-1}, y_{2n},a, b, t), M(y_{2n}, y_{2n+1},a, b, t), M(y_{2n-1}, y_{2n},a, b, t),$

$M(y_{2n}, y_{2n+1},a, b, t), M(y_{2n-1}, y_{2n+1},a, b, (1 + q)t)\})$

$\geq r(\min\{M(y_{2n-1}, y_{2n},a, b, t), M(y_{2n}, y_{2n+1}, a ,b, t), M(y_{2n-1}, y_{2n},a , b, t),$

$M(y_{2n-1}, y_{2n+1},a, b, qt)\})$

$\geq M(y_{2n-1}, y_{2n},a, b, t) * M(y_{2n}, y_{2n+1},a, b, t) * M(y_{2n-1}, y_{2n},a, b,qt)$

Since t-norm * is continuous, letting $q \to 1$, we have

$M(y_{2n}, y_{2n+1},a, b, t) \geq r(\min\{M(y_{2n-1}, y_{2n},a,b, t), M(y_{2n}, y_{2n+1},a, b, t), M(y_{2n}, y_{2n+1},a,b, t)\}$

$\geq r(\min\{M(y_{2n-1}, y_{2n},a,b, t), M(y_{2n}, y_{2n+1},a,b, t)\})$.
It follows that, $M(y_{2n}, y_{2n+1},a,b, t) > M(y_{2n-1}, y_{2n},a, b, t)$, since r(t) > t for each 0 < t < 1.

Similarly, $M(y_{2n+1}, y_{2n+2},a,b, t) > M(y_{2n}, y_{2n+1},a,b, t)$.

Therefore, in general, we have $M(y_n, y_{n+1},a, b, t) \geq r(M(y_{n-1}, y_n,a,b, t)) > M(y_{n-1}, y_n, a, b, t)$

Therefore, $\{M(y_n, y_{n+1},a,b, t)\}$ is an increasing sequence of positive real numbers in [0, 1] and tends to a limit , say $\lambda \leq 1$. We claim that $\lambda = 1$. We have $\lambda < 1$, then $M(y_n, y_{n+1},a,b, t) \geq r(M(y_{n-1}, y_n,a, b, t))$.

So on letting $n \to \infty$, we get $\lim_{n\to\infty} M(y_n, y_{n+1},a,b, t))) \geq r(\lim_{n\to\infty} M(y_n, y_{n+1},a,b,t)$

That is, $\lambda \geq r(\lambda) > \lambda$, a contradiction. Thus, we have $\lambda = 1$.
Now, for ant positive integer p, such that

$M(y_n, y_{n+p},a,b,t) \geq M(y_n, y_{n+1},a,b, t) * M(y_{n+1}, y_{n+2},a,b, t/p) * ... * M(y_{n+p-1}, y_{n+p},a,b, t/p)$.

Latting $n \to \infty$, we get $\lim_{n\to\infty} M(y_n, y_{n+p},a, b, t) \geq 1*1*1*....1 = 1$

Thus we have $\lim_{n\to\infty} M(y_n, y_{n+p}, a, b,t) = 1$. Hence, $\{y_n\}$ is a Cauchy sequence in X. Since X is complete metric space, so the sequence $\{y_n\}$ converges to a point u (say) in X and consequently, the subsequences $\{Ax_{2n}\}$, $\{Sx_{2n}\}$, $\{Tx_{2n+1}\}$ and $\{Bx_{2n+1}\}$ also converges to u.

We first consider the case when (A, S) are reciprocally continuous semi-compatible maps and (B, T) is weakly compatible. Since A and S are reciprocally continuous semi-compatible mappings, so we have $ASx_{2n} \to Au$, $SAx_{2n} \to Su$ and $M(ASx_{2n}, Su, a, b, t) = 1$. Therefore, we get Au = Su.
We claim that Au = u. For this, suppose that Au $\neq$ u.

Then, setting x = u and y = x_{2n+1} in contractive condition (ii) with $\alpha = 1$, we get

$M(Au, Bx_{2n+1}, a, b, t) \geq r(\min\{M(Au, Su, a, b, t), M(Bx_{2n+1}, Tx_{2n+1}, a, b, t), M(Su, Tx_{2n+1}, a, b, t),$

$M(Au, Tx_{2n+1}, a, b, t), M(Su, Bx_{2n+1}, a, b, t)\})$.
Letting $n \to \infty$, we get $M(Au, u, t) \geq r(M(Au, u, t)) > M(Au, u, t)$, which implies that u = Au.

Thus, we have u = Au = Su. Since $AX \subseteq TX$, so there exists v in X such that u = Au = Tv.

Therefore, setting $x = x_{2n}$ and $y = v$ in contractive condition (4.1.2) with $\alpha = 1$, we get
$M(Ax_{2n}, Bv, a, b, t) \geq r(\min\{M(Ax_{2n}, Sx_{2n}, a, b, t), M(Bv, Tv, a, b, t), M(Sx_{2n}, Tv, a, b, t),$

$M(Ax_{2n}, Tv, a, B, t), M(Sx_{2n}, Bv, a, b, t)\})$.
Letting $n \to \infty$, we get $M(Au, Bv, a, b, t) \geq r(M(Au, Bv, a, b, t)) > M(Au, Bv, a, b, t)$, which implies that $u = Bv$. Thus, we have $u = Bv = Tv$. Therefore, we get $u = Au = Su = Bv = Tv$.

Now, since $u = Bv = Tv$, so by the weak compatibility of (B, T), it follows that $BTv = TBv$ and so we get $Bu = BTv = TBv = Tu$. Thus, from the contractive condition (4.1.2) with $\alpha = 1$, we have

$M(Au, Bu, a, b, t) \geq r(\min\{M(Au, Su, a, b, t), M(Bu, Tu, a, b, t), M(Su, Tu, a, b, t),$

$M(Au, Tu, a, b, t), M(Su, Bu, a, b, t)\})$,
that is, $M(u, Bu, a, b, t) > M(u, Bu, a, b, t)$, which is a contradiction. This implies that $u = Bu$.

Similarly, using condition (4.1.2) with $\alpha = 1$, one can show that $Au = u$. Therefore, we have $u = Au = Bu = Tu = Su$. Hence, the point u is a common fixed point of A, B, S and T.
Again, we consider the case when (B, T) are reciprocally continuous semi-compatible maps and (A, S) is weakly compatible Since B and T are reciprocally continuous semi-compatible mappings, so we have $BTx_{2n} \to Bu$, $Tx_{2n} \to Tu$
We claim that $Bu = u$. For this, suppose that $Bu \neq u$. Then, setting $x = x_{2n}$ and $y = u$ in contractive condition (4.1.2) with $\alpha = 1$, we get

$M(Ax_{2n}, Bu, a, b, t) \geq r(\min\{M(Ax_{2n}, Sx_{2n}, a, b, t), M(Bu, Tu, a, b, t), M(Sx_{2n}, Tu, a, b, t),$

$M(Ax_{2n}, Tu, a, b, t), M(Sx_{2n}, Bu, a, b, t)\})$.
Letting $n \to \infty$, we get $M(u, Bu, a, b, t) \geq r(M(u, Bu, a, b, t)) > M(u, Bu, a, b, t)$, which implies that $u = Bu$.
Thus, we have $u = Bu = Tu$. Since $BX \subseteq SX$, so there exists w in X such that $u = Bu = Sw$.

Therefore, setting $x = w$ and $y = x_{2n+1}$ in contractive condition (4.1.2) with $\alpha = 1$, we get

$M(Aw, Bx_{2n+1}, a, b, t) \geq r(\min\{M(Aw, Sw, a, b, t), M(Bx_{2n+1}, Tx_{2n+1}, a, b, t), M(Sw, Tx_{2n+1}, a, b, t),$

$M(Aw, Tx_{2n+1}, a, b, t), M(Sw, Bx_{2n+1}, a, b, t)\})$.
Letting $n \to \infty$, we get $M(Aw, Bu, a, b, t) \geq r(M(Aw, Bu, a, b, t)) > M(Aw, Bu, a, b, t)$, which implies that $u = Aw$. Thus, we have $u = Aw = Sw$. Therefore, we have $u = Aw = Sw = Bu = Tu$.

Now, since $u = Aw = Sw$, so by the weak compatibility of (A, S), it follows that $ASw = SAw$ and so we get $Au = ASw = SAw = Su$. Thus, from the contractive condition (4.1.2) with $\alpha = 1$, we have
$M(Au, Bu, a, b, t) \geq r(\min\{M(Au, Su, a, b, t), M(Bu, Tu, a, b, t), M(Su, Tu, a, b, t),$

$M(Au, Tu, a, b, t), M(Su, Bu, a, b, t)\})$,
that is, $M(Au, u, a, b, t) \geq r(M(Au, u, a, b, t)) > M(Au, u, a, b, t)$, which is a contradiction.

This implies that $Au = u$. Similarly, using (4.1.2) with $\alpha = 1$, one can show that $Su = u$. Therefore, we have $u = Au = Bu = Tu = Su$. Hence, the point u is a common fixed point of A, B, S and T.

Uniqueness

The uniqueness of a common fixed point of the mappings A, B, S and T be easily verified by using (4.1.2). In fact, if u_0 be another fixed point for mappings A, B, S and T. Then, for $\alpha = 1$, we have
$M(u, u_0, a, b, t) = M(Au, Bu_0, a, b, t) \geq r(\min\{M(Au, Su, a, b, t), M(Bu_0, Tu_0, a, b, t),$

$M(Su, Tu_0, a, b, t), M(Au, Tu_0, a, b, t),$
$M(Su, Bu_0, a, b, t)\}), \geq r(M(u, u_0, a, b, t)) > M(u, u_0, a. B. t)$, and hence, we get $u = u_0$.
This completes the proof of the theorem.

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