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Research Article

THE GENERALIZED HANKEL-CLIFFORD TRANSFORMATION OF CERTAIN SPACES FOR A CLASS OF ULTRADISTRIBUTIONS

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The present paper aims to study theory for generalized Hankel-Clifford transformation on certain spaces of generalizedultradifferentiable functions. To extend the transform to a space of ultradistributions, the class of rapid descent ultra-differentiable functions are stated. Mappings involving various differential operators are investigated and stated to be continuous. The theory developed is applied to solve some partial differential equations involving generalized Kepsinki-type-operator with ultradistributional initial conditions.

INTRODUCTION

Malgonde [7] investigated the variant of the generalized Hankel-Clifford transform defined by

$$(h_{\alpha,\beta} f)(y) = F(y) = \int_0^\infty (y/x)^{-(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\sqrt{xy}\right) f(x)dx \ , \ (\alpha-\beta) \geq -1/2$$

$$= y^{-\alpha-\beta} \int_0^\infty \mathrm{J}_{\alpha,\beta}(xy) f(x)dx \ , \ (\alpha-\beta) \geq -1/2 \qquad (1)$$

where $\mathrm{J}_{\alpha,\beta}(x) = (x)^{(\alpha+\beta)/2} J_{\alpha-\beta}\left(2\sqrt{x}\right)$, $J_{\alpha-\beta}(x)$ being the Bessel function of the first kind of order $(\alpha-\beta)$, in spaces of generalized functions. Note that (1) reduces to well-known Hankel-Clifford transform for suitable values of the parameters viz. for $\alpha = 0$ and $\beta = -\mu$, a transform studied in [9].

In order to use themodified method used over the previous natural method developed by Zemanian [6] in his research on a variety of distributional series expansions. Recall, that the success of Zemanian's method lies in the fact that the differential operators considered are always selfadjoint.

In this paper some spaces of ultradifferentiable functions and their duals are developed. The generalized Hankel-Clifford transforms is a continuous linear mapping on these spaces. Therefore the transformation is linear and continuous mapping on the corresponding dual spaces. We prove certain properties of the spaces ${}^{p}S_{\beta}^{b,B}$ and study on the operatorThe operator $\Delta_{\alpha,\beta} = x^{-\beta} DP_{\alpha,\beta}$; where $D = \frac{d}{dx}; D_{\beta} = x^{-\beta} D; P_{\alpha,\beta} = x^{\alpha-\beta+1} Dx^{-\alpha}$, and $B_{\alpha,\beta}$ act on them. Certain spaces of multipliers are defined.

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Some spaces of testing functions and their duals

The space $^{p}S_{\beta,a,A}$

Let $a>0$ an arbitrary constant, $\alpha-\beta\in R$ and $p\in N$. Define the function of space $^{p}S_{\beta,a,A}$ as the collection of all complex-valued smooth functions ψ defined on $I(0,\infty)$ such that set of all infinitely smooth functions satisfying

$$\left|x^{m}D^{q}\left(x^{\beta}\psi(x)\right)\right|\leq C_{q,\delta}\left(A+\delta\right)^{m}(pm)!^{a}$$

for every $q\in N$ and $\delta>0$. $C_{q,\delta}$ are positive constant depending on ψ .

$^{p}S_{\beta,a,A}$ is a linear space with the usual operations. Moreover, if

$$\|\psi\|_{q,\delta}=\sup_{\substack{x\in I\\ m\in N}}\frac{\left|x^{m}D^{q}\left(x^{\beta}\psi(x)\right)\right|}{\left(A+\delta\right)^{m}(pm)!^{a}}$$

for every $q\in N$ and $\delta>0$, each $\|\ \|_{q,\delta}$ is a seminorm on $^{p}S_{\beta,a,A}$ and thecollection $\Gamma=\left\{\|\ \|_{q,\delta}\right\}_{q\in N,\delta>0}$ is a multinorm because each $\|\ \|_{0,\delta}$ is a norm. Since the systems of seminorms Γ and $\Gamma_{1}=\left\{\|\ \|_{q,1/n}\right\}_{q,n\in N}$ are equivalent, the space $^{p}S_{\alpha,\beta,a,A}$ equipped with the topology generated by Γ_{1} , is a countable multinormed space in [5].

Properties of the space $^{p}S_{\beta,a,A}$:

$^{p}S_{\beta,a,A}\subset H_{\beta}$, the inclusion being continuous.

$^{p}S_{\beta,a,A}$is complete and therefore aFréchet space.

Thus H_{β} is a complete space. $^{p}S_{\beta,a,A}$is a spaceof testing functions. Its dual $\left({}^{p}S_{\beta,a,A}\right)'$ is a space of generalized functions.

If $a>0$, then $D(I)\subset {}^{p}S_{\beta,a,A}$ and the topology of $D(I)$ is stronger that the topology induced by $^{p}S_{\beta,a,A}$ in $D(I)$ analogous to [2].

Proof. If $\psi\in D(I)$ one has $\left|x^{m}D^{q}\left(x^{\beta}\psi(x)\right)\right|\leq C_{q}\left(A+\delta\right)^{m}(pm)!^{a}L^{m}\left(A+\delta\right)^{-m}(pm)!^{-a}$ for every $m,q\in N$ and $a>0$, where $L=\sup\{x:x\in\operatorname{supp}\psi\}$ and $C_{q}=\sup_{0<x<L}\left|D^{q}\left(x^{\beta}\psi(x)\right)\right|$.

Hence

$\left|x^{m}D^{q}\left(x^{\beta}\psi(x)\right)\right|\leq C_{\delta}C_{q}\left(A+\delta\right)^{m}(pm)!^{a}$ with $C_{\delta}>L^{m}\left(A+\delta\right)^{-m}(pm)!^{-a}$ for $m\in N$. Consequently $D(I)\subset {}^{p}S_{\beta,a,A}$. The non-triviality of $^{p}S_{\beta,a,A}$ follows provided that $a>0$. This space is dense in $E(I)$.

On the other hand, where $a=0$ and $\sup_{x\in I}\left|x^{m}D^{q}\left(x^{\beta}\psi(x)\right)\right|\leq C_{q,\delta}\left(A+\delta\right)^{m}$ for $\delta>0$ and $m,q\in N$ then $\psi\in {}^{p}S_{\beta,a,A}$.

1. $^{p}S_{\beta,a,A}\subset H_{\beta}^{p^{a},Ap^{p^{a}}}$
2. If $p>1, H_{\beta,a,p}\subset {}^{p}S_{\beta,a,A}$
3. $H_{\beta,a,p}\subset {}^{1}S_{\beta,r_{a},A}$ with $r>1$.

All inclusions are continuous. Every inclusion transforms bounded sets into bounded sets, therefore it is continuous as in [1].

$^{p}S_{\beta+k,a,A}$ is contained in $^{p}S_{\beta,a,A}$ for each $k\in N$, then the inclusions is continuous.

Proof.Assume $k=1$ and choose $\psi={}^{p}S_{\beta+1,a,A}$. One then has

$$\sup_{x\in I}\left|x^m D^q\left(x^\beta\psi(x)\right)\right| \le \sup_{x\in I}\left|x^{m+1}D^q\left(x^{\beta-1}\psi(x)\right)\right| + q\sup_{x\in I}\left|x^m D^{q-1}\left(x^{\beta-1}\psi(x)\right)\right|$$
$$\le C_{q,\delta}\left(A+\delta\right)^{m+1} + qC_{q-1,\delta}\left(A+\delta\right)^m$$

for $\delta > 0, m \in N$ and $q \in N - \{0\}$. Also

$$\sup_{x\in I}\left|x^m D^q\left(x^\beta\psi(x)\right)\right| = \sup_{x\in I}\left|x^{m+1}D^q\left(x^{\beta-1}\psi(x)\right)\right| < C_{0,\delta}\left(A+\delta\right)^{m+1}.$$

The proof is completed by induction on k.

The following results is used to define a countable union space.

If $0 < A_1 < A_2$, then ${}^pS_{\beta,a,A_1} \subset {}^pS_{\beta,a,A_2}$ the inclusions being continuous.

Hence the union space is defined as

$${}^pS_{\beta,a} = \bigcup_{A=1}^{\infty} {}^pS_{\beta,a,A}$$

indicates inductive limit topology. ${}^pS_{\beta,a}$ is a space of testing functions and its dual, $\left({}^pS_{\beta,a}\right)'$ is a space of generalized functions.

The space ${}^pS_\beta^{b,B}$

Let $B > 0, b \ge 0$ an arbitrary constant $\alpha - \beta \in R$. Define the function of space ${}^pS_\beta^{b,B}$ as the collection of all complex-valued smooth functions $\psi(x)$ defined on $I(0,\infty)$ such that set of all infinitely smooth functions satisfying

$$\sup_{x\in I}\left|x^m D^q\left(x^\beta\psi(x)\right)\right| \le C_{m,\rho}\left(B+\rho\right)^q\left(pq\right)!^b$$

for every $m, q \in N$ and $\rho > 0$. $C_{m,\rho}$ are positive constant depending on ψ.

${}^pS_\beta^{b,B}$ is a linear space with the usual operations. Moreover, if

$$\|\psi\|^{m,\rho} = \sup_{\substack{x\in I\\ q\in N}} \frac{\left|x^m D^q\left(x^\beta\psi(x)\right)\right|}{\left(B+\rho\right)^q\left(pq\right)!^b}$$

for every $m \in N$ and $\rho > 0$, each $\|\ \|^{m,\rho}$ is a seminorm on ${}^pS_\beta^{b,B}$ and thecollection $\Gamma = \left\{\|\ \|^{m,\rho}\right\}_{m\in N,\,\rho>0}$ is a multinorm because each $\|\ \|^{0,\rho}$ is a norm. Since the systems of seminorms Γ and $\Gamma_1 = \left\{\|\ \|^{m,1/n}\right\}_{m,n\in N}$ are equivalent, the space ${}^pS_\beta^{b,B}$ equipped with the topology generated by Γ_1, is a countable multinormed space.

Properties of the space ${}^pS_\beta^{b,B}$:

1. ${}^pS_\beta^{b,B} \subset H_\beta$, the inclusion being continuous.
2. ${}^pS_\beta^{b,B}$ is complete and therefore aFréchet space.

Thus H_β is a complete space. ${}^pS_\beta^{b,B}$ is a spaceof testing functions. Its dual $\left({}^pS_\beta^{b,B}\right)'$ is a space of generalized functions.

If $b > 0$, then $\boldsymbol{D}(I) \subset {}^pS_\beta^{b,B}$ and the topology of $\boldsymbol{D}(I)$ is stronger that the topology induced by ${}^pS_\beta^{b,B}$ in $\boldsymbol{D}(I)$.

Proof. If $\psi \in \boldsymbol{D}(I)$ one has $\left|x^m D^q\left(x^\beta\psi(x)\right)\right| \le C_{m,\rho}\left(B+\rho\right)^m$

for every $m, q \in N$ and $\rho > 0$. Hence

$\left|x^m D^q\left(x^\beta \psi(x)\right)\right| \le C_{m,\rho}(B+\rho)^q (pq)!^b$ for $q \ge q_0$. Consequently $\boldsymbol{D}(I) \subset {}^p S_\beta^{b,B}$.

On the other hand, where $b=0$ and $\sup_{x\in I}\left|x^m D^q\left(x^\beta \psi(x)\right)\right| \le C_{m,\rho}(B+\rho)^m$ for $\rho>0$ and $m,q\in N$ then $\psi \in {}^p S_\beta^{b,B}$.

1. ${}^p S_\beta^{b,B} \subset H_\beta^{p^b, Bp^{p^b}}$
2. If $p>1, H_\beta^{b,\beta} \subset {}^p S_\beta^{b,B}$
3. $H_\beta^{b,\beta} \subset {}^1 S_\beta^{r_b,B}$ with $r>1$.

All inclusions are continuous. Every inclusion transforms bounded sets into bounded sets, therefore it is continuous.

Test of convergence in ${}^p S_\beta^{b,B}$

Let $\{\psi_\nu\}_{\nu\in N}$ be a sequence. If a positive constant $C_{m,\rho}$ exists for any $m\in N$ and $\rho>0$ such that $\|\psi_\nu\|^{m,\rho} < C_{m,\rho}$ for every $\nu \in N$.

$D^q\left(x^\beta \psi_\nu(x)\right) \to 0$ as $\nu\to\infty$ uniformly on $x\in(0,\varepsilon)$ for every $q\in N$ and $\varepsilon>0$, then $\psi_\nu \to 0$ as $\nu\to\infty$ in ${}^p S_\beta^{b,B}$

Proof.Let $m\in N$ and $\rho,\eta>0$. Choose ρ' such that $0<\rho'<\rho$. In these conditions:

$\|\psi_\nu\|^{m,\rho} < C_{m,\rho'} \ne 0$ for every $\nu\in N$ where $C_{m,\rho'}$ is a constant. There exists $q_0\in N$ such that $\left(\frac{B+\rho'}{B+\rho}\right)^q < \frac{\eta}{C_{m,q}}$ for every $q>q_0$.

$\left|x^m D^q\left(x^\beta \psi_\nu(x)\right)\right| \le C_{m,\rho'}(B+\rho')^q (pq)!^b < \eta(B+\rho)^q (pq)!^b$ for every $q>q_0$.

By taking $q<q_0$ and $x>C_{m+1,\rho/\eta}$,

$$\left|x^m D^q\left(x^\beta \psi_\nu(x)\right)\right| = \frac{\left|x^{m+1} D^q\left(x^\beta \psi_\nu(x)\right)\right|}{x} \le \frac{1}{x}\|\psi_\nu\|^{m+1,\rho}(B+\rho)^q (pq)!^b < \eta(B+\rho)^q (pq)!^b.$$

By virtue of uniform convergence [1], there exists $\nu_0\in N$ such that

$\left|x^m D^q\left(x^\beta \psi_\nu(x)\right)\right| \le \eta(B+\rho)^q (pq)!^b$.

For $\nu\ge\nu_0, q<q_0$ and $x<C_{m+1,\rho/\eta}$,

$\left|x^m D^q\left(x^\beta \psi_\nu(x)\right)\right| \le \eta(B+\rho)^q (pq)!^b$.

For $\nu\ge\nu_0, q\in N$ and $x\in I$,

$\left|x^m D^q\left(x^\beta \psi_\nu(x)\right)\right| \le \eta(B+\rho)^q (pq)!^b$.

In other words:

$\|\psi_\nu\|^{m,\rho} \le \eta$ for $\nu\ge\nu_0$. Then $\psi_\nu\to 0$ as $\nu\to\infty$ in ${}^p S_\beta^{b,B}$.

${}^p S_{\beta+k}^{b,B}$ is contained in ${}^p S_\beta^{b,B}$ for each $k\in N$, then the inclusions is continuous.

Proof.Assume $k=1$ and choose $\psi = {}^p S_{\beta+k}^{b,B}$. One then has

$$\begin{aligned}\sup_{x\in I}\left|x^m D^q\left(x^\beta \psi(x)\right)\right| &\le \sup_{x\in I}\left|x^{m+1} D^q\left(x^{\beta-1}\psi(x)\right)\right| + \sup_{x\in I}\left|x^m D^{q-1}\left(x^{\beta-1}\psi(x)\right)\right| \\ &\le C_{m,\rho'}(B+\rho)^{m+1} + qC_{m-1,\rho}(B+\rho)^m\end{aligned}$$

for $\rho>0, m\in N$ and $q\in N-\{0\}$. Also

$$\sup_{x\in I}\left|x^m D^q\left(x^{\beta}\psi(x)\right)\right| = \sup_{x\in I}\left|x^{m+1} D^q\left(x^{\beta-1}\psi(x)\right)\right| < C_{0,\rho}\left(B+\rho\right)^{m+1}.$$

The proof is completed by induction on k.

The following results is used to define a countable union space.

If $0 < B_1 < B_2$, then $S_{\beta}^{b,B_1} \subset {}^{p}S_{\beta}^{b,B_2}$ the inclusions being continuous.
Hence the union space is defined as

$${}^{p}S_{\beta}^{b} = \bigcup_{B=1}^{\infty} {}^{p}S_{\beta}^{b,B}$$

indicates inductive limit topology. ${}^{p}S_{\beta}^{b}$ is a space of testing functions and its dual, $\left({}^{p}S_{\beta}^{b}\right)'$ is a space of generalized functions.

The space ${}^{p}S_{\beta,a,A}^{b,B}$

Let $A, B > 0$ and $a, b \geq 0$ an arbitrary constant $\alpha - \beta \in R$. Define the function of space ${}^{p}S_{\beta,a,A}^{b,B}$ as the collection of all complex-valued smooth functions $\psi(x)$ defined on $I(0,\infty)$ such that set of all infinitely smooth functions satisfying

$$\sup_{x\in I}\left|x^m D^q\left(x^{\beta}\psi(x)\right)\right| \leq C_{\delta,\rho}\left(A+\delta\right)^m (pm)!^a \left(B+\rho\right)^q (pq)!^b$$

for every $m, q \in N$ and $\delta, \rho > 0$. $C_{\delta,\rho}$ are positive constant depending on ψ.
${}^{p}S_{\beta,a,A}^{b,B}$ is a linear space with the usual operations. Moreover, if

$$\left\|\psi\right\|_{\delta}^{\rho} = \sup_{\substack{x\in I\\ m\in N\\ q\in N}} \frac{\left|x^m D^q\left(x^{\beta}\psi(x)\right)\right|}{\left(A+\delta\right)^m (pm)!^a \left(B+\rho\right)^q (pq)!^b}$$

for every $m, q \in N$ and $\delta, \rho > 0$, each $\|\ \|_{\delta}^{\rho}$ is a seminorm on ${}^{p}S_{\beta,a,A}^{b,B}$ and thecollection $\Gamma_2 = \left\{\|\ \|_{\delta}^{\rho}\right\}_{\delta,\ \rho>0}$ is a multinorm because each $\|\ \|^{0,\rho}$ is a norm. Since the systems of seminorms Γ_2 and $\Gamma_3 = \left\{\|\ \|_{1/n}^{1/n}\right\}_{n\in N}$ are equivalent, the space ${}^{p}S_{\beta,a,A}^{b,B}$ equipped with the topology generated by Γ_3, is a countable multinormed space.
Properties of the space ${}^{p}S_{\beta,a,A}^{b,B}$:
${}^{p}S_{\beta,a,A}^{b,B} \subset H_{\beta}$, the inclusion being continuous.
${}^{p}S_{\beta,a,A}^{b,B}$ is complete and therefore aFréchet space.

Thus H_{β} is a complete space. ${}^{p}S_{\beta,a,A}^{b,B}$ is a spaceof testing functions. Its dual $\left({}^{p}S_{\beta,a,A}^{b,B}\right)'$ is a space of generalized functions.

If $a, b > 0$, then $\boldsymbol{D}(I) \subset {}^{p}S_{\beta,a,A}^{b,B}$ and the topology of $\boldsymbol{D}(I)$ is stronger that the topology induced by ${}^{p}S_{\beta,a,A}^{b,B}$ in $\boldsymbol{D}(I)$.

Proof. If $\psi \in \boldsymbol{D}(I)$ for every $m, q \in N$ and $\delta, \rho > 0$. Hence
$\sup_{x\in I}\left|x^m D^q\left(x^{\beta}\psi(x)\right)\right| \leq C_{\delta,\rho}\left(A+\delta\right)^m (pm)!^a \left(B+\rho\right)^q (pq)!^b$ for $q \geq q_0$. Consequently $\boldsymbol{D}(I) \subset {}^{p}S_{\beta,a,A}^{b,B}$.
On the other hand, where $b = 0$ and $\sup_{x\in I}\left|x^m D^q\left(x^{\beta}\psi(x)\right)\right| \leq C_{\delta,\rho}\left(A+\delta\right)^m (pm)!^a \left(B+\rho\right)^q (pq)!^b$ for $\delta, \rho > 0$ and $m, q \in N$ then $\psi \in {}^{p}S_{\beta,a,A}^{b,B}$.

1. ${}^{p}S_{\beta,a,A}^{b,B} \subset H_{\beta,p^a,Ap^{p^a}}^{p^b,Bp^{p^b}}$
2. If $p > 1$, $H_{\beta,a,A}^{b,B} \subset {}^{p}S_{\beta,a,A}^{b,B}$
3. $H_{\beta,a,A}^{b,B} \subset {}^{1}S_{\beta,r_a,A}^{r_b,B}$ with $r > 1$.

All inclusions are continuous. Every inclusion transforms bounded sets into bounded sets, therefore it is continuous.

Test of convergence in ${}^{p}S^{b,B}_{\beta,a,A}$

Let $\{\psi_\nu\}_{\nu\in N}$ be a sequence. If a positive constant $C_{\rho,\delta}$ exists for any $m,n\in N$ and $\rho,\delta>0$ such that $\|\psi_\nu\|^{\rho}_{\delta}<C_{\delta,\rho}$ for every $\nu\in N$.

$D^q\left(x^\beta\psi_\nu(x)\right)\to 0$ as $\nu\to\infty$ uniformly on $x\in(0,\varepsilon)$ for every $m,q\in N$ and $\varepsilon>0$, then $\psi_\nu\to 0$ as $\nu\to\infty$ in ${}^{p}S^{b,B}_{\beta,a,A}$.

Proof.Let $m,n\in N$ and $\delta,\ \rho>0$. Choose δ',ρ' such that $0<\delta'<\delta$ and $0<\rho'<\rho$. In these conditions:

$\|\psi_\nu\|^{\rho}_{\delta}<C_{\delta',\rho'}\neq 0$ for every $\nu\in N$ where $C_{\delta',\rho'}$ is a constant. There exists $m_0,q_0\in N$ such that $\left(\dfrac{B+\rho'}{A+\delta'}\right)^q<\dfrac{\lambda}{C_{m,q}}$ for every $m>m_0,\ q>q_0$.

$\left|x^mD^q\left(x^\beta\psi_\nu(x)\right)\right|\le C_{\delta',\rho'}(A+\delta')^m(pm)!^a(B+\rho')^q(pq)!^b<\lambda(A+\delta)^m(pm)!^a(B+\rho)^q(pq)!^b$ for every $m>m_0,\ q>q_0$.

By taking $q<q_0$ and $x>C_{m+1,\delta/\lambda,\rho/\lambda}$,

$$\begin{aligned}\left|x^mD^q\left(x^\beta\psi_\nu(x)\right)\right|&=\frac{\left|x^{m+1}D^q\left(x^\beta\psi_\nu(x)\right)\right|}{x}\\&\le\frac{1}{x}\|\psi_\nu\|^{m+1,\rho,\delta}\lambda(A+\delta)^m(pm)!^a(B+\rho)^q(pq)!^b\\&<\lambda(A+\delta)^m(pm)!^a(B+\rho)^q(pq)!^b.\end{aligned}$$

By virtue of uniform convergence, there exists $\nu_0\in N$ such that

$\left|x^mD^q\left(x^\beta\psi_\nu(x)\right)\right|\le\lambda(A+\delta)^m(pm)!^a(B+\rho)^q(pq)!^b$.

For $\nu\ge\nu_0,q<q_0$ and $x<C_{m+1,\rho/\eta}$,

$\left|x^mD^q\left(x^\beta\psi_\nu(x)\right)\right|\le\lambda(A+\delta)^m(pm)!^a(B+\rho)^q(pq)!^b$.

For $\nu\ge\nu_0,q\in N$ and $x\in I$,

$\left|x^mD^q\left(x^\beta\psi_\nu(x)\right)\right|\le\lambda(A+\delta)^m(pm)!^a(B+\rho)^q(pq)!^b$.

In other words:

$\|\psi_\nu\|^{\rho}_{\delta}\le\lambda$ for $\nu\ge\nu_0$. Then $\psi_\nu\to 0$ as $\nu\to\infty$ in ${}^{p}S^{b,B}_{\beta,a,A}$.

${}^{p}S^{b,B}_{\beta+k,a,A}$ is contained in ${}^{p}S^{b,B}_{\beta,a,A}$ for each $k\in N$, then the inclusions is continuous.

Proof.Assume $k=1$ and choose $\psi={}^{p}S^{b,B}_{\beta+k,a,A}$. One then has

$$\begin{aligned}\sup_{x\in I}\left|x^mD^q\left(x^\beta\psi(x)\right)\right|&\le\sup_{x\in I}\left|x^{m+1}D^q\left(x^{\beta-1}\psi(x)\right)\right|+\sup_{x\in I}\left|x^mD^{q-1}\left(x^{\beta-1}\psi(x)\right)\right|\\&\le C_{m+1,\delta',\rho'}(B+\rho)^{m+1}(A+\delta)^{m+1}+qC_{m-1,\rho,\delta}(B+\rho)^m(A+\delta)^m\end{aligned}$$

for $\delta,\rho>0,m,n\in N$ and $q\in N-\{0\}$. Also

$$\sup_{x\in I}\left|x^mD^q\left(x^\beta\psi(x)\right)\right|=\sup_{x\in I}\left|x^{m+1}D^q\left(x^{\beta-1}\psi(x)\right)\right|<C_{0,\delta,\rho}(A+\delta)^{m+1}(B+\rho)^{m+1}.$$

The proof is completed by induction on k.

The following results is used to define a countable union space.

If $0 < A_1 < A_2$ and $0 < B_1 < B_2$, then ${}^pS^{b,B_1}_{\beta,a,A_1} \subset {}^pS^{b,B_2}_{\beta,a,A_2}$ the inclusions being continuous.
Hence the union space is defined as

$${}^pS^b_{\beta,a} = \bigcup_{B=1}^{\infty} {}^pS^{b,B}_{\beta,a,A}$$

indicates inductive limit topology. ${}^pS^b_{\beta,a}$ is a space of testing functions and its dual, $\left({}^pS^b_{\beta,a}\right)'$ is a space of generalized functions.

The nontriviality of spaces ${}^pS^b_{\beta,a}$

The mapping $S^b_{\beta,a} \to H^b_{\beta,a}$ holds true. So $\psi(y) \to y^\beta \psi(y)$ is linear and continuous. The properties of the spaces of type $S^b_{\beta,a}$ are nontrivial.

1. ${}^pS_{\beta,a,A}$ and ${}^pS^{b,B}$ for every $a,b \geq 0$, $A,B > 0$,
2. ${}^pS^{b,B}_{\beta,a,A}$ for $a \geq 1$ and $b = 0$ or $a = 0$ and $b > 1$: $A,B > 0$,
3. ${}^pS^{b,B}_{\beta,a,A}$ for $A,B > 0$ and $a + b \geq 1$,
4. ${}^pS^{b,B}_{\beta,a,A}$ for $p > 1$, $a,b > 0$, $A,B > \gamma$ and $a + b = 1$ where γ is a positive constant.

Operational Calculus

Some important linear differential operators are defined in this section. And also shown that they are continuous on the previously introduced spaces.

Property The mapping $x^n : {}^pS^b_{\beta,a} \to {}^pS^b_{\beta,a}$ is linear and continuous for every $n \in \mathbb{N}$.

Proof. Assuming $n = 1$ and considering $\psi \in {}^pS^{b,B}_{\beta,a,A}$,

$$\begin{aligned}
\left|x^m D^q\left(x^\beta \psi(x)\right)\right| &\leq \left|x^{m+1} D^q\left(x^{\beta-1}\psi(x)\right)\right| + q\left|x^m D^{q-1}\left(x^{\beta-1}\psi(x)\right)\right| \\
&\leq C_{\delta,\rho}\begin{Bmatrix}(A+\delta)^{m+1}\left(p(m+1)\right)!^a (B+\rho)^q (pq)!^b \\ +q(A+\delta)^m (pm)!^a (B+\rho)^{q-1}\left(p(q-1)\right)!^b\end{Bmatrix} \\
&\leq C'_{\delta,\rho}(A+\delta)^m (pm)!^a (B+\rho)^q (pq)!^b
\end{aligned}$$

for every $m \in \mathbb{N}, q \in \mathbb{N} - \{0\}$ and $\delta, \rho > 0$ in virtue of [2].
For $q = 0$; for $m \in \mathbb{N}$ and $\delta, \rho > 0$ above relation becomes:
$\left|x^m\left(x^\beta \psi(x)\right)\right| \leq C'_{\delta,\rho}(A+\delta)^m (pm)!^a$.

Hence $x\psi \in {}^pS^{b,B}_{\beta,a,A}$ and the mapping $x^n : {}^pS^b_{\beta,a} \to {}^pS^b_{\beta,a}$ is linear and continuous. The proof is completed by induction.

The procedure is analogous in any of the spaces under consideration.
Property Let l be a real number. Denoting ${}^pS^b_{\beta,a}$ any of the spaces ${}^pS_{\beta,a,A}$, ${}^pS^{b,B}_{\beta}$, ${}^pS^{b,B}_{\beta,a,A}$ or the respective union spaces, then the operator $x^l : {}^pS^b_{\beta,a} \to {}^pS^b_{\beta-1,a}$ is an isomorphism.

Property The differential operator R_β is an isomorphism from ${}^pS^b_{\beta,a}$ into ${}^pS^b_{\beta-1,a}$; its inverse being its inverse beingas in [8],

$$R_\beta^{-1} = x^{-\beta}\int_\infty^x t^{\beta-1}\phi(t)\,dt, \phi \in {}^pS^b_{\beta-1,a}.$$

Proof.Operator R_β and its inverse are linear. If $\psi \in {}^pS^{b,B}_{\beta,a,A}$, then

$$\left|x^m D^q\left(x^{\beta-1}\psi(x)\right)\right| \le \left|x^{m+1}D^{q-1}\left(x^{\beta}\psi(x)\right)\right|$$
$$\le C_{\delta,\rho}\left(A+\delta\right)^m (pm)!^a \left(B+\rho\right)^q (pq)!^b$$

for every $m,q \in \mathbb{N}$ and $\delta,\rho > 0$. Hence R_β is a continuous mapping.

For $q>0$,

$$\left|x^m D^q\left(x^{\beta}R_\beta^{-1}\psi(x)\right)\right| \le \left|x^m D^q\left(\int_\infty^x x^{\beta-1}\psi(t)\,dt\right)\right|$$
$$\le C_{\delta,\rho}\left(A+\delta\right)^m (pm)!^a \left(B+\rho\right)^q (pq)!^b.$$

If $q=0$, one has:

$$\left|x^m\left(x^{\beta}R_\beta^{-1}\psi(x)\right)\right| \le \left|x^m\left(\int_\infty^x x^{\beta-1}\psi(t)\,dt\right)\right| < C_{\delta,\rho}\left(A+\delta\right)^m (pm)!^a.$$

Hence $R_\beta^{-1}\psi \in {}^pS_{\beta,a,A}^{b,B}$ and R_β^{-1} is a continuous operator. The proof is similar in case of ${}^pS_\beta^{b,B}$ and ${}^pS_{\beta,a,A}^{b,B}$.

Property Let $D: {}^pS_{\beta,a}^{b} \to {}^pS_{\beta-1,a}^{b}$ is linear and continuous.

ProofLet $\psi \in {}^pS_{\beta,a,A}^{b,B}$. Then

$$\left|x^m D^q\left(x^{\beta+1}\psi(x)\right)\right| \le \left|x^m D^q\left(x^{\beta-1}\psi(x)\right)(-\beta+q)\right| + \left|x^m D^{q+1}\left(x^{\beta}\psi(x)\right)\right|$$
$$\le C_{\delta,\rho}\left(A+\delta\right)^m (pm)!^a \left(B+\rho\right)^q (pq)!^b$$

for every $m,q \in \mathbb{N}$ and $\delta,\rho > 0$.

PropertyThe operator $B_\beta = DR_\beta$ from ${}^pS_{\beta,a}^{b}$ into itself is linear and continuous. Defining the generalized $D^*, R_\beta^*, R_\beta^{-1*}$ and B_β^* as the adjoint of the classical operators D, R_β, R_β^{-1} and B_β respectively. Then

Property 3.6. The operators $D^*:\left({}^pS_{\beta-1,a}^{b}\right)' \to \left({}^pS_{\beta,a}^{b}\right)'$ and $B_\beta^*:\left({}^pS_{\beta,a}^{b}\right)' \to \left({}^pS_{\beta,a}^{b}\right)'$ are linear and continuous.

The mapping $R_\beta^*:\left({}^pS_{\beta-1,a}^{b}\right)' \to \left({}^pS_{\beta,a}^{b}\right)'$ is an isomorphism R_β^{-1*} is its inverse.

Multipliers in Spaces of type ${}^pS_{\beta,a}^{b}$

For smooth functions on $0<x<\infty$ which are multipliers of type ${}^pS_{\beta,a}^{b}$:

$$\left|x^m D^k\left(x^{\beta}\theta(x)\psi(x)\right)\right| \le C_k^{\beta,\delta}\left(A+\delta\right)_{m,k} a_{m,k} \qquad (1)$$

where $A, C_k^{\beta,\delta}$ are positive constants depending on $\theta(x)\psi(x)$ and $a>0$ being an arbitrary constant.

Thus $\theta(x)\psi(x)$ is in ${}_pS_{\beta,A}$ and the mapping $h_\beta: {}_pS_{\beta,A} \to {}_pS_\beta^A, \psi \to \theta\psi$, is continuous.

Taking a ψ in ${}_pS_{\beta,A}$

$$\left|x^m D^k\left(x^{\beta}\psi(x)\theta(x)\right)\right| \le C_k^{\beta,\rho}\left(B+\rho\right)_{m,k} b_{m,k} \qquad (2)$$

where $B, C_k^{\beta,\rho}$ are positive constants depending on $\theta(x)\psi(x)$ and $b>0$ being an arbitrary constant.

Considering from [8], $\mathrm{J}_{\alpha,\beta}(z) = z^{(\alpha+\beta)/2}J_{\alpha-\beta}\left(2\sqrt{z}\right)$. And as the transformation is an automorphism onto $H_{\alpha,\beta}$ for $\alpha-\beta$, $\frac{d^n}{dz^n}\mathrm{J}_{\alpha,\beta}(z) = (-1)^n\mathrm{J}_{\alpha,\beta+n}(z),\ n\in\mathbb{N}$, then for every $\phi \in H_{\alpha,\beta}$ and $m,k \in \mathbb{N}$ from[4],

$$y^m D^k\left(y^{\beta}\psi(y)\right)=(-1)^k\int_0^{\infty}(xy)^m \mathrm{J}_{\alpha,\beta+k+m}(xy)\,x^{(-\beta+k+m)}D^m x^{\beta}\phi(x)\,dx \tag{3}$$

where $\psi(y)=\hbar_{\alpha,\beta}\{\phi(x)\}(y)$.

The virtue of boundedness of the function $z^m \mathrm{J}_{\alpha,\beta+k+m}(z)$, (3) is given by

$$\left|y^m D^k\left(y^{\beta}\psi(y)\right)\right|\le M\sup_{x\in I}\left|x^{(c+k+m)}D^m x^{\beta}\phi(x)\right| \tag{4}$$

for $m,k\in\mathbb{N}$ and $\mu\ge 0$, being $c=[-\beta]$ and M is a constant.

To study the image of ${}_pS_{\beta}$ by h_{β}. Let ϕ be any element of ${}_pS_{\beta,A}$ invoking (4), then

$$\left|y^m D^k\left(y^{\beta}\psi(y)\right)\right|\le K_{m,\delta}(A+\delta)^k\left\{p\left((c+k+m)\right)\right\}!^a\le C_{m,\delta}(A+\delta)^k(pk)!^a \tag{5}$$

for every $m,k\in\mathbb{N}$ and δ>0.

Hence the mapping $h_{\beta}:{}_pS_{\beta,A}\to{}_pS_{\beta}^{A}$ is linear and continuous.

If $\phi\in{}_pS_{\alpha,B}$, then

$$\left|y^m D^k\left(y^{-\alpha}\psi(y)\right)\right|\le M_{k,\rho}\left\{C_{c+k+m,\rho}(B+\rho)^m mp\right\}!^b \tag{6}$$

for $m,k\in\mathbb{N}$ and ρ>0.

Therefore the mappings:

$h_{\alpha}:{}_pS_{\alpha,\mathrm{B}}\to{}_pS_{\alpha}^{B}$ is linear and continuous.

If $\phi\in{}_pS_{\alpha,\beta}^{A,B}$, then

$$\begin{aligned}\left|y^m D^k\left(y^{-(\alpha-\beta)}\psi(y)\right)\right|&\le C_{m,\delta}(A+\delta)^k(pk)!^a\times M_{k,\rho}\left\{C_{c+k+m,\rho}(B+\rho)^m mp\right\}!^b\\&\le M_{\delta,\rho}\left(Ae^{p^{\alpha}}+\eta\right)^k\left(Be^{p^{\alpha}}+\varepsilon\right)^m(p)!^{a+b}(k)!^a(m)!^b\end{aligned} \tag{7}$$

for $m,k\in\mathbb{N}$ and η,ε>0.

Thus it has been established that the mapping $h_{\alpha,\beta}:{}_pS_{\alpha,\beta}^{A,B}\to{}_pS_{\alpha,\beta}^{Ae^{p^{\alpha}},Be^{p^{\alpha}}}$ is linear and continuous.

Application

Now considering the initial value problem analogous to [3],

$$\frac{\partial u(x,t)}{\partial t}=P\left(B_{\alpha}^{*}\right)u(x,t) \tag{8}$$

$u(x,0)=u_0(x)$ with $u_0\in\varphi'$.

The generalized Hankel-Clifford transformation $h_{\alpha}^{*\prime}$, leads to the new equivalent problem

$$\frac{\partial v(y,t)}{\partial t}=P(-y)v(y,t) \tag{9}$$

$v(y,0)=v_0(y)$

where $v(y,t)=h_{\alpha}^{*\prime}\{u(x,t),x\to y\}$ and $v_0(y)=h_{\alpha}^{*\prime}\{u_0(x)\}(t)$. (10)

A formal solution of (3) is the generalized function $v(y,t)=v_0(y)e^{-yt}$.

The distribution $u(x,t) = \hbar'_{\alpha,\beta}\left\{v_0(y)e^{-yt}; y \to x\right\} \in \phi'$ is a solution of (1).

Accordingly one has:

$$\frac{\partial}{\partial t}\left\langle \hbar'_{\alpha,\beta}\left\{v_0(y)e^{-yt}\right\}, \phi\right\rangle = \frac{\partial}{\partial t}\left\langle u_0, \hbar_{\alpha,\beta}\left\{e^{-yt}\hbar_{\alpha,\beta}\{\phi\}\right\}\right\rangle$$
$$= \left\langle u_0, \hbar_{\alpha,\beta}\left\{e^{-yt}\hbar_{\alpha,\beta}\left\{P\left(\Delta_{\alpha,\beta}\right)\right\}\right\}\right\rangle$$
$$= \left\langle P\left(\Delta^*_{\alpha,\beta}\right)\hbar'_{\alpha,\beta}e^{-yt}\hbar'_{\alpha,\beta}u_0, \phi\right\rangle.$$

for every $\phi \in \varphi$ and

$$\left\langle \hbar'_{\alpha,\beta}\left\{e^{-yt}\hbar'_{\alpha,\beta}\{u_0\}\right\}, \phi\right\rangle = \left\langle u_0, \hbar_{\alpha,\beta}\left\{e^{-yt}\hbar_{\alpha,\beta}\{\phi\}\right\}\right\rangle \to \langle u_0, \phi\rangle$$

for every $\phi \in \varphi$.

Thus arriving to the following theorem as:

Theorem: The generalized function $u(x,t) = \hbar'_{\alpha,\beta}\left\{e^{-yt}\hbar'_{\alpha,\beta}\{u_0\}\right\} \in \phi'$ is a solution of (8) for every initial value $u_0 \in \varphi'_4$.

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