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Research Article

AN EFFICIENT ALGORITHM TO EVALUATE THE HANKEL TRANSFORM USING B-POLYNOMIAL MULTI-WAVELETS

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A R T I C L E I N F O

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A B S T R A C T

A numerical method for evaluating the Hankel transform of order of function is given. The method is based upon B-polynomial multi-wavelets forms an orthonormal bases for we expand the part of the integrand in its wavelet series reducing the Hankel transform integral reduces a series of Bessel function multiplied by the wavelet coefficients of the input function. The proposed method has been illustrated by examples.

INTRODUCTION

The Hankel transform is a very useful tool in broad area of physical problems which have an axial symmetry [1]. There are two types of the Hankel transform. The first one is defined on the semi-infinite interval. In this case Hankel transform and inverse Hankel transforms are defined by

$$F_n(p) = \int_0^\infty t\ f(t)\, J_n(pt)\, dt, \tag{1}$$

and

$$f(t) = \int_0^\infty p\, F_n(p)\, J_n(pt)\, dp, \tag{2}$$

where J_n is the n-th order Bessel function of the first kind [4]. In the case of the finite Hankel transform only a direct transform has an integral form, without loss of generality. It is defined by

$$\hat{F}_n(p) = \int_0^1 t f(t)\, J_n(pt) dt\,. \tag{3}$$

The Hankel transform is also useful in geophysics and Cosmology, for example [3, 4]. In some cases, analytical evaluations are rare and then numerical methods have becomeimportant. The usual classical methods are Trapezoidal rule, Cotes rule etc. in all these methods the integrand is generally replaced by a sequence of polynomials; which are more accurate, if integrand is smooth. But in this paper, I have used $t\ f(t)\, J_n(pt)$ and $p\, F_n(p)\, J_n(pt)$ which are rapidly oscillating functions for large t and p, respectively. These difficulties can be solved by different techniques. First, the fast Hankel transform is proposed by Siegman and Huang *et al*. in [2, 8]. The second method is based on the use of Filon quadrature philosophy [6]. In Filon quadrature philosophy, the integrand is product of two components (assume), first is slowly varying component and second is rapidly oscillating component. In the case of the Hankel transform, the former is $t\ f(t)$ and the latter is $J_n(pt)$. Later in 2003 Postnikov [3], Zykov and Postnikov [9] proposed, for the first time a novel and powerful method for computing zero and first order Hankel transform by using Haar bases (orthogonal) and piecewise-linear bases (non orthogonal), respectively.More, recently, Singh *et al*. [10] give another powerful method, for solving the Hankel transform by using linear Legendre multiwavelets. In this paper, we present a method that is very accurate and fast for numerical evaluation of Hankel transform using B-polynomials multiwavelets. Numerical evaluations of test functions with known analytical Hankel transforms illustrate the method.

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B-Polynomial Multiwavelets: Wavelets as a family of functions of constructed from translation and dilation of a single function · , called the mother wavelet, we define wavelets by

$$\cdot_{b,a}(t)=\frac{1}{\sqrt{|a|}}\cdot\left(\frac{t-b}{a}\right), \qquad a,b\in R, \quad a\neq 0, \qquad (4)$$

where a is called a scaling parameter which measures the degree of scale, and b is a translation parameter which determines the time location of the wavelet. When parameters a and b vary continuously then wavelets called continuous wavelets. If we restrict the parameter a and b to discrete values as $a=a_0^{-k}$, $b=n\,b_0\,a_0^{-k}$ $\;a_0>1,\; b_0>0$, and n, and k positive integers. We fix two positive constants a_0 and b_0, define discrete wavelets:

$$\cdot_{k,n}(t)=|a_0|^{k/2}\cdot\left(a_0^k\,t-n\,b_0\right), \qquad (5)$$

where $\cdot_{k,n}(t)$ form a wavelets basis for $L^2(R)$. In particular, when $a_0=2,\; and\; b_0=1$ then $\cdot_{k,n}(t)$ forms an orthonormal basis [5, 9 and 11].

In general the B-polynomials of m[th] degree are defined on the interval [0, 1) as [7, 11]

$$B_{i,m}(t)=\frac{m!}{i!\,(m-i)!}\,x^i(1-x)^{m-i}, \qquad 0\le i\le m\,. \quad (6)$$

It can be easily shown that the B-polynomials is positive and also the sum of all the B-polynomials is unity for all real x belonging to the interval [0, 1) i.e.

$$\sum_{i=0}^{m} B_{i,m}(t)=1\,. \qquad (7)$$

We can use $B_{i,m}(t)$ as orthonormal basis and expand any polynomial of degree m in linear combination

$$P(t)=\sum_{i=0}^{m} c_i\,B_{i,m}(t), \qquad m\ge 1\,. \quad (8)$$

The B-polynomials can also define on the interval [0, 1] by using recursive definition of $B_{i,m}(t)$. The B-polynomial multi-wavelets $\cdot_{n,m}(t)=\cdot\,(k,n,m,t)$ have four ***arguments:*** translation argument $n=0,1,2,...,2^k-1,$ dilation argument *k* can assume any positive integer, *m* is the order for B-polynomial and *t* is the normalized time [11]. They are defined on the interval [0, 1) as

$$\cdot_{n,m}(t)=\begin{cases}2^{k/2}WB_m(2^k t-n), & for\,\frac{n}{2^k}\le t\le\frac{n+1}{2^k}\\ 0, & otherwise\end{cases} \qquad (9)$$

where $m=0,1,...,M,\qquad n=0,1,2,...,2^k-1.$ In equation (2.6) the coefficient $2^{k/2}$ is for orthonormality, the dilation parameter is $a=2^{-k}$ and translation parameter is $b=n\,2^{-k}$. Here, $WB_m(t)$ is the orthonormal form of B-polynomials of order m.The following Lemma which has been purposed by [11] showed an upper bound to estimate the error.

Lemma

Suppose that the function $f:[0,1]\to R$ is *m* times continuously differentiable, $f\in C^m[0,1]$. Then $C^T\Psi$ approximate f with mean error bounded as follows:

$$\left\|f-C^T\Psi\right\|\le\frac{1}{m!\,2^{mk}}\sup_{x\in[0,1]}\left|f^{(m)}(x)\right|. \quad (10)$$

As can be seen that by Lemma 2.1 the upper bound of the error depends upon the factor $1/m!2^{mk}$ which shows that the error rapidly tend to zero as *m* and *k* increase slowly. Note that in the classical orthogonal basis such as Fourier, Legendre, Chebyshev, etc, the upper bound of the error depends on $1/m!$. This is the most advantage of the new technique.

Derivation of the Method

In this section, we give derivation of the method of Hankel transform. Consider the function $f(t)$ having compactly supported. This function $f(t)$ represent physical fields, which is zero outside a disk with finite radius, say R. In cases where the function $f(t)$ is not compact, we assume that given $\in>0$ there exists a compact interval $I_\in$ such that $|f(t)|<\in\;\; for\; x\notin I_\in$. Hence it is more appropriate to consider the finite Hankel transform. Suppose $\sup(f)\subset[0,h]$ then (1) written by

$$\hat{F}_n(p)=\int_0^1 t\,f(t)\,J_n(pt)\,dt \quad ,(11)$$

known as the finite Hankel transform (FHT), where *t* is replace by *t/h*. writing $t\,f(t)=g(t)$ in equation (11), we get

$$\hat{F}_n(p)=\int_0^1 g(t)\,J_n(pt)\,dt\;. \qquad (12)$$

As $g(t)\in L^2[0,1]$,

$$g(t)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{nm}\cdot_{nm}(t), \qquad (13)$$

where

$c_{nm}=(f(t),\cdot_{nm}(t))$, in which (., .) denote the inner product.

If the infinite series in (13) is truncated, then equation can be written as

$$g(t)\cong\sum_{n=0}^{2^k-1}\sum_{m=0}^{M} c_{nm}\cdot_{nm}(t)=C^T\,\Psi(t)\,, \qquad (14)$$

where C and $\Psi(t)$ are $(2^k-1)(M+1)\times 1$ matrices given by

$$C=[c_{00},c_{01},...,c_{0M},c_{10},...,c_{1M},...,c_{(2^k-1)0},...,c_{(2^k-1)M}]^T,$$

$$\Psi=[\cdot_{00}(t),\cdot_{01}(t),...;_{0M}(t),\cdot_{10}(t),...;_{1M}(t),...;_{(2^k-1)0}(t),...;_{(2^k-1)M}(t)]^T.$$

From equations (12) and (14), we get

$$\hat{F}_n(p) \cong \int_0^1 C^T \Psi(t)\, J_n(pt)\,dt \qquad (15)$$

$$= C^T \int_0^1 \Psi(t)\, J_n(pt)\,dt .$$

Illustrative examples

We applied the method presented in this paper and solved three examples given in [10].

Example

The circc function is very useful that can be defined as

$$Circ\ (t/a) = \begin{cases} 1, & t \le a \\ 0, & t \ge a \end{cases} . \qquad (16)$$

The zeroth-order Hankel transform of $Circ(t/a)$ is the Sombrero function [2], given by

$$S_0(p) = \frac{a^2 J_1(ap)}{ap} . \qquad (17)$$

Putting a = 1, we have

$$S_0(p) = \frac{J_1(p)}{p} . \qquad (18)$$

Since in this case B-polynomial multi-wavelets series representation (13) at level $M = 1$ *and* $k = 0$ we obtain

$$g(t) = c_{00} \cdot {}_{00}(t) + c_{01} \cdot {}_{01}(t) = \frac{1}{\sqrt{3}} \cdot {}_{00}(t) + 0 \cdot {}_{01}(t) .$$

Therefore, from (15) we get the exact solution

$$\hat{F}_n(p) = \int_0^1 t\, J_n(pt)\,dt = \frac{J_1(p)}{p}$$

Example

Taking $f(t) = \sqrt{1-t^2},\ 0 \le t \le 1,$ then

$$F_1(p) = \frac{® J_1(p/2)}{2p} . \qquad (19)$$

The truncated B-polynomial multi-wavelets series representation (15) gives the approximate solution of (19) at level $M = 5, k = 0$ we study that over finite interval [0, 1], equation (15) is approximate for $F_1(p)$ in fig.1 we show that the exact transform $F_1(p)$ (solid line), and the transform $\hat{F}_1(p)$

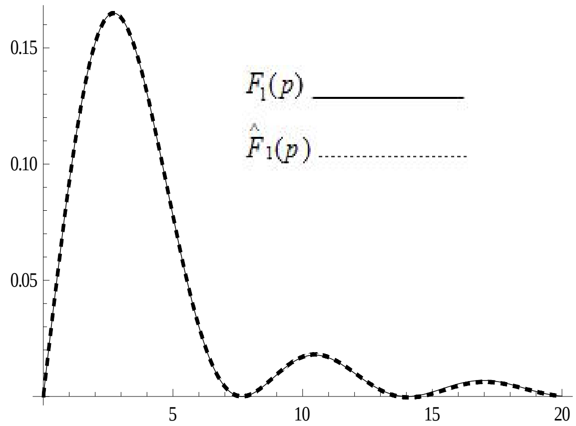

Figure 1 The exact transform, $F_1(p)$ (solid line) and the approximation transform, $\hat{F}_1(p)$ (dotted-line) truncated at a level $M = 5, k = 0.$

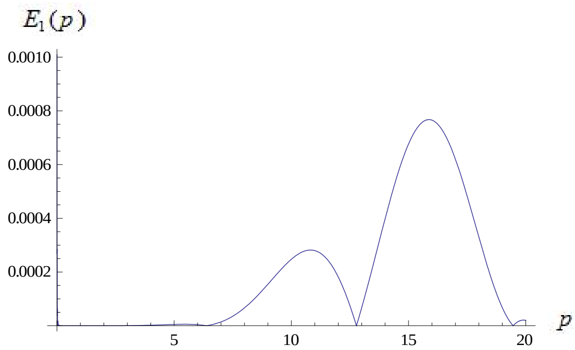

Figure 2. Absolute error between the exact transform $F_1(p)$ and the approximation transform, $\hat{F}_1(p)$ truncated at a level $M = 5, k = 0.$

Example

In this example, we choose as a test function the generalized version of the top-hat function, gives as $f(t) = t^{¬}\,[H(t) - H(t-a)], a > 0$ and $H(t)$ is the step unit function $H(t) = \begin{cases} 1, & t \ge 0 \\ 0, & t < 0 \end{cases}.$

Then

$$F_{¬}(p) = \frac{J_{¬+1}(p)}{p} .$$

We take $a = 1, ¬ = \frac{1}{10}, 5$ and observe that the errors are quite small as shown in Fig.3 and Fig.4 respectively.

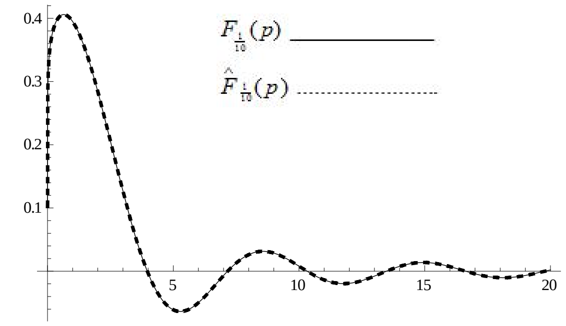

Figure 3 The exact transform $F_{\frac{1}{10}}(p)$ (solid line) and the approximation transform, $\hat{F}_{\frac{1}{10}}(p)$ (dotted-line) truncated at a level $M = 5, k = 0, \neg = \frac{1}{10}$.

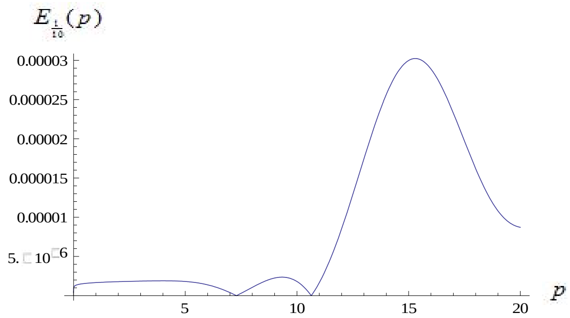

Figure 4 Absolute error between the exact transform $F_{\frac{1}{10}}(p)$ and the approximation transform, $\hat{F}_{\frac{1}{10}}(p)$ truncated at a level $M = 5, k = 0, \neg = \frac{1}{10}$.

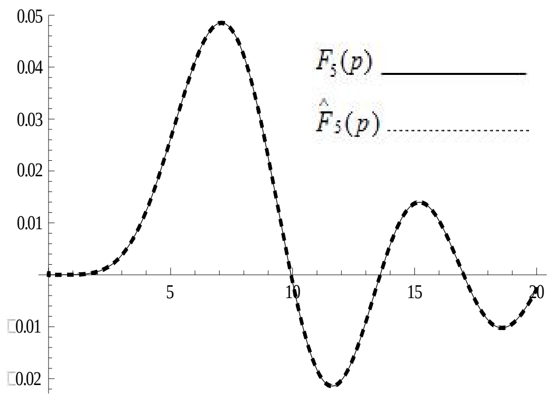

Figure 5. The exact transform, $F_5(p)$ (solid line) and the approximation transform, $\hat{F}_5(p)$ (dotted-line) truncated at a level $M = 5, k = 0, \neg = 5$.

CONCLUSION

The aim of the present work, solving Hankel transform having compact support using B-polynomial multiwavelets [11], it makes them more useful and simple in actual computations. Our choice of B-polynomial multiwavelets makes its more attractive in application in applied physical problems. It give better approximation compared to that of Postnikov [3], Zykov & Postinikov [9] and Singh *et al*. [10].

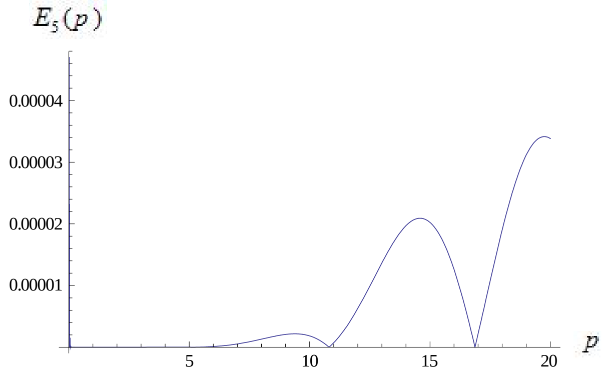

Figure 6. Absolute error between the exact transform $F_5(p)$ and the approximation transform, $\hat{F}_5(p)$ truncated at a level $M = 5, k = 0, \neg = 5$.

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