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Research Article

ON g#s*-HOMEOMORPHISM IN TOPOLOGICAL SPACES

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ABSTRACT

In this paper we introduce a new class of closed maps namely g#s-closed maps also introduce a new class of homeomorphisms called g#s*-homeomorphisms and prove that the set of all g#s*-homeomorphisms form a group under the operation composition of maps.

INTRODUCTION

The notion homeomorphism plays an important role in topology. A homeomorphism is a bijective map f: X → Y when both f and f^{-1} are continuous. Veera kumar [5] in 2002 introduced the concept of $g^{\#}$-semi-closed sets in topological spaces. In this paper we first introduce a new class of closed maps namely $g^{\#}$s-closed maps and then we introduce and study $g^{\#}$s*-homeomorphisms in a topological space. We also prove that the set of all $g^{\#}$s*-homeomorphisms form a group under the operation of composition of maps.

Preliminaries

Throughout this paper (X, τ), (Y, σ) and (Z, η) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of space (X, τ) the cl(A), int(A) and A^c denote the closure of A, the interior of A and the complement of A in X respectively.
We recall the following definitions:

Definition: A subset A of a topological space (X, τ) is called semi-open [1] (resp. semi- closed [1]) if A ⊆ cl(int(A)) (resp. int(cl(A)) ⊆ A).

The semi-closure [3] of a subset A of X (denoted by scl(A)) is defined to be the intersection of all semi-closed sets containing A.

Definition: A subset A of a topological space (X, τ) is called α-open [2] (resp. α-closed [2]) if A ⊆ int(cl(int(A))) (resp. cl(int(cl(A))).

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The α-closure of a subset A of X (denoted by αcl(A)) is defined to be the intersection of all α-closed sets containing A.

Definition: A subset A of a topological space (X, τ) is called

1. αg-closed [4] if αcl(A) ⊆ U whenever A ⊆ U and U is open in X. The complement of αg-closed set is called αg-open.
2. $g^{\#}$s-closed [5] if scl(A) ⊆ U whenever A ⊆ U and U is αg-open in X. The complement of $g^{\#}$s-closed set is called $g^{\#}$s-open.

Definition: A function f: (X, τ) → (Y, σ) is called

1. $g^{\#}$s-continuous [5] if the inverse image of every σ-closed set in Y is $g^{\#}$s-closed in X.
2. $g^{\#}$s-irresolute [5] if the inverse image of every $g^{\#}$s-closed set in Y is $g^{\#}$s-closed in X.

3.0 $g^{\#}$s-Closed Maps

In this section we introduce the following definitions.

Definition: A map f: (X, τ) → (Y, σ) is called g#s-closed (resp. g#s-open) map if f(A) is g#s-closed (resp. g#s-open) set in (Y, σ) for every closed (open) set A of (X, τ).

Definition: Let (X, τ) be a topological space and A ⊆ X. We define the g#s-interior of A (briefly g#s-int(A)) to be the union of all g#s-open sets contained in A.

Definition: Let (X, τ) be a topological space and A ⊆ X. We define the $g^{\#}$s-closure of A (briefly $g^{\#}$s-cl(A)) is defined as the intersection of all $g^{\#}$s-closed sets containing A i.e. $g^{\#}$s-cl(A) =

$\cap$ {B: A $\subseteq$ B and B $\in$ $G^{\#}SC$ (X, τ)}. Here $G^{\#}SC$ represent family of $g^{\#}$s-closed sets.

Theorem: Let (X, τ) be a topological space and A $\subseteq$ X then following properties are follows:

1. $g^{\#}$s-cl(A) is the smallest $g^{\#}$s-closed set containing A.
2. A is $g^{\#}$s-closed iff $g^{\#}$s-cl(A) = A.

Proof: Follows from definitions.

Theorem: For any two subsets A and B of (X, τ)

1. If A $\subseteq$ B then $g^{\#}$s-cl(A) $\subseteq$ $g^{\#}$s-cl(B).
2. $g^{\#}$s-cl(A $\cap$ B) $\subseteq$ $g^{\#}$s-cl(A) $\cap$ $g^{\#}$s-cl(B).

Proof: Immediately follows from definitions.

Theorem: If B $\subseteq$ A $\subseteq$ X, B is a $g^{\#}$s-closed set relative to A and A is open and $g^{\#}$s-closed in (X, τ). Then B is $g^{\#}$s-closed in (X, τ).

Corollary: If A is $g^{\#}$s-closed set and B is closed set then A $\cap$ B is $g^{\#}$s-closed set.
Proof: Follows immediately since every closed set is *gs-closed.

Definition: Let (X, τ) be a topological space and A $\subseteq$ X then We define $g^{\#}$s-interior of A (briefly $g^{\#}$s-int(A)) as the union of all $g^{\#}$s-open sets contained in A.
Lemma: For any A $\subseteq$ X, int(A) $\subseteq$ $g^{\#}$s-int(A) $\subseteq$ A.
Proof: Since every open set is $g^{\#}$s-open so proof straight forward.

Definition: A map f: (X, τ) $\rightarrow$ (Y, σ) is called $g^{\#}$s-closed (resp. $g^{\#}$s-open) if the image of every closed (resp. open) set in (X, τ) is $g^{\#}$s-closed (resp. $g^{\#}$s-open) in (Y, σ).

Theorem: A map f: (X, τ) $\rightarrow$ (Y, σ) is $g^{\#}$s-closed iff $g^{\#}$s-cl(f(A)) $\subseteq$ f(cl(A)) for every subset A of (X, τ).
Proof: Follows from theorem (3.02) and (3.03).

Theorem: A map f: (X, τ) $\rightarrow$ (Y, σ) is $g^{\#}$s-closed iff for each subset A of (Y, σ) and for each open set U containing f^{-1}(A) there exists a $g^{\#}$s-open set V of (Y, σ) such that A $\subseteq$ V and f^{-1}(V) $\subseteq$ U.

Proof: Let f is $g^{\#}$s-closed map. Let A $\subseteq$ Y and U be an open subset of (X, τ) such that f^{-1}(A) $\subseteq$ U then V = $(f(U^C))^C$ is a $g^{\#}$s-open set containing A such that f^{-1}(V) $\subseteq$ U.

Conversely let A be a closed set in X then $f^{-1}((f(A))^C) \subseteq A^C$ and A^C is open in X. By assumption there exists a $g^{\#}$s-open set V of (Y, σ) s.t. $(f(A))^C \subseteq V$ and $f^{-1}(V) \subseteq A^C$ so $A \subseteq (f^{-1}(V))^C$. Hence $V^C \subseteq f(A) \subseteq f((f^{-1}(V))^C) \subseteq V^C$ i.e. f(A) = V^C, since V^C is $g^{\#}$s-closed so f(A) is $g^{\#}$s-closed i.e. f is $g^{\#}$s-closed map.

Remark: The following example shows that the composition of two $g^{\#}$s-closed maps need not be $g^{\#}$s-closed.

Example: Let X = Y = Z = {a, b, c}, τ = {ϕ, {a}, {a, b}, X}, σ = {ϕ, {a}, Y}, η = {ϕ, {a}, {b, c}, Z}. Define f: (X, τ) $\rightarrow$ (Y, σ) and g: (Y, σ) $\rightarrow$ (Z, η) by identity mapping then f and g both are $g^{\#}$s-closed map but their composition gof: (X, τ) $\rightarrow$ (Z, η) is not a $g^{\#}$s-closed map.

Theorem: Let f: (X, τ) $\rightarrow$ (Y, σ) be a closed map and g: (Y, σ) $\rightarrow$ (Z, η) be a $g^{\#}$s-closed map then their composition gof: (X, τ) $\rightarrow$ (Z, η) is $g^{\#}$s-closed.

Remark: If f: (X, τ) $\rightarrow$ (Y, σ) is $g^{\#}$s-closed and g: (Y, σ) $\rightarrow$ (Z, η) is closed then their composition gof: (X, τ) $\rightarrow$ (Z, η) need not be a $g^{\#}$s-closed map as seen from the following example.

Example: Let X = Y = Z = {a, b, c}, τ = {ϕ, {a}, {c}, {a, b}, {a, c}, X}, σ = {ϕ, {a}, {c}, {a, c}, Y}, η = {ϕ, {b}, {c}, {b, c}, {a, c}, Z}. Define f: (X, τ) $\rightarrow$ (Y, σ) byidentity mapping and g: (Y, σ) $\rightarrow$ (Z, η) by g(a) = b, g(b) = a, g(c) = c. Then f is $g^{\#}$s-closed map and g is closed map but their composition gof: (X, τ) $\rightarrow$ (Z, η) is not a $g^{\#}$s-closed map.

Theorem: Let f: (X, τ) $\rightarrow$ (Y, σ) and g: (Y, σ) $\rightarrow$ (Z, η) be two mappings such that their composition gof: (X, τ) $\rightarrow$ (Z, η) be a $g^{\#}$s-closed map then the following statements are true.

1. If f is continuous and serjective then g is $g^{\#}$s-closed map.
2. If g is $g^{\#}$s-irresolute and injective then f is $g^{\#}$s-closed map.

Theorem: Let f_A be the restriction of a map f: (X, τ) $\rightarrow$ (Y, σ) to a subset A of (X, τ) then

1. If f: (X, τ) $\rightarrow$ (Y, σ) is $g^{\#}$s-closed and A is a closed subset of (X, τ), then f_A: (A, τ_A) $\rightarrow$ (Y, σ) is $g^{\#}$s-closed.
2. If f: (X, τ) $\rightarrow$ (Y, σ) is $g^{\#}$s-closed (resp. closed) and A = f^{-1}(B) for some closed (resp. $g^{\#}$s-closed) set B of (Y, σ) then f_A: (A, τ_A) $\rightarrow$ (Y, σ) is $g^{\#}$s-closed.

Proof: Let B be a closed set of A. Then B = A $\cap$ C for some closed set C of (X, τ) and so B is closed in (X, τ). By hypothesis, f(B) is $g^{\#}$s-closed in (Y, σ). But f(B) = f_A(B) and so f_A is a $g^{\#}$s-closed map.
Let D be a closed set of A. Then D = A $\cap$ E for some closed set E in (X, τ). Now f_A(D) = f(D) = f(A $\cap$ E) = f(f^{-1}(B) $\cap$ E) = B $\cap$ f(E). Since f is $g^{\#}$s-closed, f(E) is $g^{\#}$s-closed and so B $\cap$ f(E) is $g^{\#}$s-closed in (Y, σ) by corollary (3.05). Thus f_A is $g^{\#}$s-closed map.

Theorem: For any bijective f: (X, τ) $\rightarrow$ (Y, σ) the following statements are equivalent.

a. f^{-1}: (Y, σ) $\rightarrow$ (X, τ) is $g^{\#}$s-continuous.
b. f is a $g^{\#}$s-open map and
c. f is a $g^{\#}$s-closed map.

Theorem: Let f: (X, τ) $\rightarrow$ (Y, σ) be a $g^{\#}$s-open map then for a subset A of (X, τ), f(int(A)) $\subseteq$ $g^{\#}$s-int(f(A)).
Theorem: A function f: (X, τ) $\rightarrow$ (Y, σ) is $g^{\#}$s-open if and only if for any subset B of (Y, σ) and for any closed set A containing f^{-1}(B), there exists a $g^{\#}$s-closed set C of (Y, σ) containing B such that f^{-1}(C) $\subseteq$ A.
Proof: Similar to theorem (2.10).

Corollary: A function f: (X, τ) $\rightarrow$ (Y, σ) is $g^{\#}$s-open if and only if f^{-1}($g^{\#}$s-cl(A)) $\subseteq$ cl(f^{-1}(A) for every subset A of (Y, σ).

Definition: A map f: (X, τ) $\rightarrow$ (Y, σ) is said to be a $g^{\#}$s*-closed (resp. $g^{\#}$s*-open) if the image f(A) is $g^{\#}$s-closed (resp. $g^{\#}$s-open) set in (Y, σ) for every $g^{\#}$s-closed (resp. $g^{\#}$s-open) set A in (X, τ).

Theorem: Every $g^{\#}$s*-closed map is $g^{\#}$s-closed map.

The converse is not true in general as it can be seen from the following example.

Example: Let X = Y = {a, b, c}, τ = {ϕ, {a}, X} and σ = {ϕ, {a}, {b, c}, Y}. Define f: (X, τ) → (Y, σ) by identity mapping then f is $g^{\#}$s-closed map but not $g^{\#}$s*-closed map.

Theorem: A map f: (X, τ) → (Y, σ) is $g^{\#}$s*-closed iff $g^{\#}$s-cl(f(A)) ⊆ f($g^{\#}$s-cl(A)) for every subset A of (X, τ).
Proof: Similar to theorem (3.09).

Theorem: For any bijection f: (X, τ) → (Y, σ) the following are equivalent

a. f^{-1}: (Y, σ) → (X, τ) is $g^{\#}$s-irresolute,
b. f is a $g^{\#}$s*-open map and
c. f is a $g^{\#}$s*-closed map.

4.0 $g^{\#}$s* -Homeomorphisms

In this section we introduce the following definitions.

Definition: *A bijection f: (X, τ) → (Y, σ) is called $g^{\#}$s*-homeomorphism if both f and f^{-1} are $g^{\#}$s*-irresolute. We denote the family of all $g^{\#}$s*-homeomorphism of a topological space (X, τ) onto itself by $g^{\#}$s*-h(X, τ).*

Theorem: If f: (X, τ) → (Y, σ) and g: (Y, σ) → (Z, η) are $g^{\#}$s*-homeomorphism then their composition gof: (X, τ) → (Z, η) is also $g^{\#}$s*-homeomorphism.

Proof: Let U be $g^{\#}$s*-open set in (Z, η) then g^{-1}(U) is $g^{\#}$s*-open set in Y and so $f^{-1}(g^{-1}(U)) = (gof)^{-1}(U)$ is $g^{\#}$s*-open in (X, τ) so gof is $g^{\#}$s*-irresolute.

Again let V be $g^{\#}$s*-open set in X then by hypothesis f(V) is $g^{\#}$s*-open in Y and then g(f(V)) = (gof)(V) is $g^{\#}$s*-open in Z so $(gof)^{-1}$ is $g^{\#}$s*-irresolute. Hence gof is a $g^{\#}$s*-homeomorphism.

Theorem: The set $g^{\#}$s*-h(X, τ) is a group under the composition of maps.

Proof: Define a binary operation ∗ : $g^{\#}$s*-h(X, τ) × $g^{\#}$s*-h(X, τ) → $g^{\#}$s*-h(X, τ) by f∗g = gof for all f and g ∈ $g^{\#}$s*-h(X, τ) and o is the usual operation of composition of maps then by theorem (4.02), gof ∈ $g^{\#}$s*-h(X, τ). Again composition of maps is associative and the identity map I: (X, τ) → (X, τ) belonging to $g^{\#}$s*-h(X, τ) is the identity element. If f ∈ $g^{\#}$s*-h(X, τ) then f^{-1} ∈ $g^{\#}$s*-h(X, τ) s.t. $fof^{-1} = f^{-1}of = I$ so inverse exist for all element of $g^{\#}$s*-h(X, τ). Thus ($g^{\#}$s*-h(X, τ), o) is a group under the operation of composition of maps.

Theorem: Let f: (X, τ) → (Y, σ) is a $g^{\#}$s*-homeomorphism then f induces an isomorphism from the group $g^{\#}$s*-h(X, τ) onto the group $g^{\#}$s*-h(Y, σ).

Proof: Define θ_f: $g^{\#}$s*-h(X, τ) → $g^{\#}$s*-h(Y, σ) by θ_f (h) = $fohof^{-1}$ for every h ∈ $g^{\#}$s*-h(X, τ). Then θ_f is a bijection. Again for all h_1, h_2 ∈ $g^{\#}$s*-h(X, τ), θ_f (h_1oh_2) = fo(h_1oh_2)of^{-1} = (foh_1of^{-1})o(foh_2of^{-1}) = θ_f (h_1)oθ_f (h_2) so θ_f is a homeomorphism and so it is an isomorphism induced by f.

Theorem: $g^{\#}$s*-homeomorphism is an equivalence relation in the collection of all topological spaces.
Proof: Follows from theorem (4.02).

Theorem: If f: (X, τ) → (Y, σ) is a $g^{\#}$s*-homeomorphism, then $g^{\#}$s-cl(f^{-1}(A)) ⊆ f^{-1}($g^{\#}$s-cl((B)) for all A ⊆ Y.

Corollary: If f: (X, τ) → (Y, σ) is a $g^{\#}$s*-homeomorphism, then $g^{\#}$s-cl(f(A)) = f($g^{\#}$s-cl((A)) for all A ⊆ X.

Corollary: If f: (X, τ) → (Y, σ) is $g^{\#}$s*-homeomorphism, then f($g^{\#}$s-int(A)) = $g^{\#}$s-int(f(A)) for all A ⊆ X.

Proof: Follows from corollary (4.07).

Corollary: If f: (X, τ) → (Y, σ) is a $g^{\#}$s*-homeomorphism, then f^{-1}($g^{\#}$s-int(A)) = $g^{\#}$s-int(f^{-1}(A)) for all A ⊆ Y.
Proof: Follows from corollary (4.08).

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