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Research Article

A GENERALIZED FIXED POINT THEOREM IN G-METRIC SPACE

Manoj Garg[1] and Shikha Agarwal[2]

[1]Durga Narayan Degree (P.G.) College, Fatehgarh, Farrukhabad (U.P.)
[2]Department of Mathematics, SCRIET, CCS University, Meerut(U.P.)

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ABSTRACT

In this paper, we prove some fixed point theorem for mappings satisfying new contractive condition in G-metric space and prove uniqueness of such fixed point also.

INTRODUCTION

In past decades the Banach fixed point theorem for contraction mapping has been generalized and extended in many directions, since it plays a major role in mathematics and applied sciences. Mustafa and Sims[10] introduce the concept of G-metric space as a generalization of metric space. The aim of this paper is to obtain a generalized common fixed point theorem for mappings satisfying contractive condition on complete G-metric space.

Preliminaries

Definition 2.1[10]: Let X be a non empty set and let G : X x X x X → R^+ be a function satisfying the following conditions :

(G_1) G(x, y, z) = 0 if x = y = z,
(G_2) G(x, x, y) > 0, for all x, y ϵ X, with x ≠ y,
(G_3) G(x, x, y) ≤ G(x, y, z), for all x, y, z ϵ X with z ≠ y,
(G_4) G(x, y, z) = G(x, z, y) = G(y, z, x) =....... (Symmetry in all three variables),
(G_5) G(x, y, z) ≤ G(x, a, a) + G(a, y, z), for all x, y, z, a ϵ X (rectangle inequality),

Then the function is called a generalized metric i.e. G-metric on X and the pair (X, G) is called a G-metric space.

Definition 2.2[10]: Let (X, G) and (X′, G′) be two G- metric spaces and let f : (X, G) → (X′, G′) be a function, then f is said to G-continuous at a point a ϵ X if given ϵ > 0 there exists δ > 0 such that x, y ϵ X; G(a, x, y) < δ implies G′(f(a), f(x), f(y)) < ϵ.

A function f ix G-continuous on X iff it is G-continuous at all a ϵ X.

Definition 2.3[10]: Let (X, G) ba a metric space, then for x_0 ϵ X, r > 0, the G- ball with centre x_0 and radius r is B_G (x_0, r) = {y ϵ X : G(x_0, y, y) < r}.

Definition 2.4[10]: Let (X, G) be a G-metric space and {x_n} be a sequence of points in X. Then {x_n} is G-convergent to x if $\lim_{n\to\infty}$ G(x, x_n, x_m) = 0, i. e., for each ϵ > 0 there exists a positive integer N such that G(x, x_n, x_m) < ϵ for all m, n ≥ N. We call that x is the limit of sequence and we write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Definition 2.5[10]: Let (X, G) be a G-metric space. A sequence {x_n} is said to be a G-cauchy sequence for each ϵ > 0 there exist a positive integer N such that G(x_n, x_m, x_{ml}) < ϵ for all l, m, n ≥ N.

Definition 2.6[10]: A G-metric space (X, G) is called a symmetric G-metric space if G(x, y, y) = G(y, x, x) for all x, y ϵ X.

Definition 2.7[10]: Let (X, G) be a G-metric space and f : X → X be a self mapping on (X, G). Then T is said to be a contraction if

G(fx, fy, fz) ≤ α G(x, y, z), for all x, y, z ϵ X where 0 ≤ α < 1.

Main Theorem

Theorem 3.1: Let (X, G) be a complete G-metric space and let f : X → X be a self mapping such that f satisfies,

$$G(fx, fy, fz) \leq \alpha G(x, fx, fx) + \beta G(y, fy, fy) + \gamma G(z, fz, fz) + \delta G(x, fy, fz) \quad (3.1)$$

for every x, y, z ϵ X and α, β, γ, δ ≥ 0 with 0 ≤ α + β + γ + 5δ < 1. Then f has a unique fixed point u in X and f is G-continuous at u.

Proof: Let x_0 ϵ X be an arbitrary point and {x_n} be a sequence such that $x_n = f^n(x_0)$,

Corresponding author:* **Manoj Garg
Durga Narayan Degree (P.G.) College, Fatehgarh, Farrukhabad (U.P.)

Then by (3.1)

$G(x_n, x_{n+1}, x_{n+1}) = G(f^n(x_0), f^{n+1}(x_0), f^{n+1}(x_0))$
$\leq \alpha\ G(f^{n-1}(x_0), f^n(x_0), f^{n+1}(x_0)) + \beta\ G(f^n(x_0), f^{n+1}(x_0), f^{n+1}(x_0))$
$+ \gamma\ G(f^n(x_0), f^{n+1}(x_0), f^{n+1}(x_0)) + \delta\ G(f^{n-1}(x_0), f^{n+1}(x_0), f^{n+1}(x_0))$
$\leq \alpha\ G(x_{n-1}, x_n, x_n) + \beta\ G(x_n, x_{n+1}, x_{n+1}) + \gamma\ G(x_n, x_{n+1}, x_{n+1}) + \delta\ G(x_{n-1}, x_{n+1}, x_{n+1})$
$\leq \alpha\ G(x_{n-1}, x_n, x_n) + \beta\ G(x_n, x_{n+1}, x_{n+1}) + \gamma\ G(x_n, x_{n+1}, x_{n+1}) + \delta\ [G(x_{n-1}, x_n, x_n) + G(x_{n+1}, x_n, x_n) + G(x_{n+1}, x_n, x_n)]$
$\leq \alpha\ G(x_{n-1}, x_n, x_n) + \beta\ G(x_n, x_{n+1}, x_{n+1}) + \gamma\ G(x_n, x_{n+1}, x_{n+1}) + \delta\ [G(x_{n-1}, x_n, x_n) + 2G(x_{n+1}, x_{n+1}, x_n) + 2G(x_{n+1}, x_{n+1}, x_n)]$

Thus we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{\alpha+\delta}{1-(\beta+\gamma+4\delta)}\ G(x_{n-1}, x_n, x_n)$$
$$\leq q\ G(x_{n-1}, x_n, x_n)$$

Where $q = \dfrac{\alpha+\delta}{1-(\beta+\gamma+4\delta)}$, and $0 < q < 1$

Moreover for all m, n ϵ N and n < m, by rectangular inequality

$G(x_n, x_m, x_m) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \ldots\ldots\ldots + G(x_{m-1}, x_m, x_m)$
$\leq (q^n + q^{n+1} + \ldots\ldots\ldots q^{m-1})\ G(x_0, x_1, x_1)$
$\leq \dfrac{q^n}{1-q}\ G(x_0, x_1, x_1)$

If m, n → ∞, then $G(x_n, x_m, x_m) = 0$ i. e. $\{x_n\}$ is a Cauchy sequence. Since X is complete there exist u ϵ X such that $\{x_n\}$ is G-converges to u.

Now if fu ≠ u then

$G(x_n, fu, fu) \leq \alpha\ G(x_{n-1}, x_n, x_n) + \beta\ G(u, fu, fu) + \gamma\ G(u, fu, fu) + \delta\ G(x_{n-1}, fu, fu)$

Since f is G-continuous, taking limit as n → ∞ we have

$G(u, fu, fu) \leq \alpha\ G(u, u, u) + (\beta + \gamma)\ G(u, fu, fu) + \delta\ G(u, fu, fu)$
$\leq (\beta + \gamma + \delta)\ G(u, fu, fu)$. This contradiction so fu = u.

For uniqueness, suppose u ≠ v, such that fv = v, then

$G(u, v, v) \leq \alpha\ G(u, fu, fu) + (\beta + \gamma)\ G(v, fv, fv) + \delta\ G(u, fv, fv)$
$\leq \alpha\ G(u, fu, fu) + (\beta + \gamma)\ G(v, fv, fv) + \delta\ G(u, fv, fv)$

Or (1-d) $G(u, v, v) \leq 0$, so that $G(u, v, v) = 0$, Which implies u = v.

For G-continuity of f, let $\{y_n\}$ be a sequence in X such that lim $y_n = u$, then

$G(u, fy_n, fy_n) \leq \alpha\ G(u, fu, fu) + \beta\ G(y_n, fy_n, fy_n) + \gamma\ G(y_n, fy_n, fy_n) + \delta\ G(u, fy_n, fy_n)$
$\leq \alpha\ G(u, fu, fu) + (\beta + \gamma)\ G(y_n, fy_n, fy_n) + \delta\ G(u, fy_n, fy_n)$
$\leq (\beta + \gamma)\ G(y_n, u, u) + (\beta + \gamma)\ G(y_n, fy_n, fy_n) + \delta\ G(u, fy_n, fy_n)$

Or $G(u, fy_n, fy_n) \leq \dfrac{\beta+\gamma}{1-(\beta+\gamma+\delta)}\ G(y_n, u, u)$

Taking limit as n → ∞, we get $G(u, fy_n, fy_n) \to 0$ i. e. $fy_n \to u = fu$.

Thus f is G-continuous at u.

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