International Journal of Current Advanced Research
ISSN: O: 2319-6475, ISSN: P: 2319 – 6505, Impact Factor: SJIF: 5.995
Available Online at www.journalijcar.org
Volume 6; Issue 3; March 2017; Page No. 2351-2357
DOI: http://dx.doi.org/10.24327/ijcar.2017.2357.0008

Research Article

ON THE LOCATION OF ZEROS OF POLYNOMIALS

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ARTICLE INFO

Article History:

Received 9th December, 2016
Received in revised form 14th January, 2017
Accepted 12th February, 2017
Published online 28th March, 2017

Key words:

Coefficients, Polynomial, Zeros.

ABSTRACT

In this paper we restrict the real and imaginary parts of the coefficients of a polynomial and find a region containing all its zeros. In addition to being generalizations of some known results, our results give many other interesting results for particular choices of the parameters.

INTRODUCTION

As for the region containing all the zeros of a polynomial with real monotonically decreasing positive coefficients, Enestrom and kakeya proved the following elegant result known as the Enestrom-Kakeya Theorem [3,4]:

Theorem A: all the zeros of a polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ satisfying $a_n \ge a_{n-1} \ge .... \ge a_1 \ge a_0 > 0$ lie in $|z| \le 1$.

Various generalizations and extensions of this result are available in the literature. Recently Gulzar [2] proved the following result:

Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j, \mathrm{Im}(a_j)=\beta_j, j=0,1,2,......,n$ such that for some $\lambda, \mu, 0 \le \lambda \le n-1, 0 \le \mu \le n-1$

and for some $k_1, k_2 \ge 1$,

$$k_1^{n-\lambda+1}\alpha_n \ge k_1^{n-\lambda}\alpha_{n-1} \ge ...... \ge k_1^2\alpha_{\lambda+1} \ge k_1\alpha_\lambda,$$
$$k_2^{n-\mu+1}\beta_n \ge k_2^{n-\mu}\beta_{n-1} \ge ...... \ge k_2^2\beta_\mu \ge k_2\beta_\mu,$$

and

$$L=|\alpha_\lambda-\alpha_{\lambda-1}|+|\alpha_{\lambda-1}-\alpha_{\lambda-2}|+......+|\alpha_1-\alpha_0|+|\alpha_0|,$$
$$M=|\beta_\mu-\beta_{\mu-1}|+|\beta_{\mu-1}-\beta_{\mu-2}|+......+|\beta_1-\beta_0|+|\beta_0|.$$

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Then all the zeros of P(z) lie in

$$\left|z+\frac{(k_1-1)\sim_n+i(k_2-1)\;_n}{a_n}\right|\le\frac{1}{|a_n|}[\sim_n+\;_n+(k_1-1)\sum_{j=\text{ª}+1}^{n}(|\sim_j|+\sim_j)-(k_2-1)\sum(|\;_j|+\;_j)$$

$$-(k_1-1)|\sim_n|-(k_2-1)|\;_n|+L+M-\sim_{\text{ª}}-\;_{\text{«}}].$$

MAIN RESULTS

In this paper we prove the following result:

Theorem 1: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\sim_j,\mathrm{Im}(a_j)=\;_j,\ j=0,1,2,......,n$ such that for some $\text{ª},\text{«},0\le\text{ª}\le n-1,0\le\text{«}\le n-1$
and for some $k_1,k_2\le 1$,

$$k_1^{\,n-\text{ª}+1}\sim_n\le k_1^{\,n-\text{ª}}\sim_{n-1}\le......\le k_1^{\,2}\sim_{\text{ª}+1}\le k_1\sim_{\text{ª}},$$
$$k_2^{\,n-\text{«}+1}\;_n\le k_2^{\,n-\text{«}}\;_{n-1}\le......\le k_2^{\,2}\;_{\text{«}}\le k_2\;_{\text{«}},$$

and

$$L=|\sim_{\text{ª}}-\sim_{\text{ª}-1}|+|\sim_{\text{ª}-1}-\sim_{\text{ª}-2}|+......+|\sim_1-\sim_0|+|\sim_0|,$$
$$M=|\;_{\text{«}}-\;_{\text{«}-1}|+|\;_{\text{«}-1}-\;_{\text{«}-2}|+......+|\;_1-\;_0|+|\;_0|.$$

Then all the zeros of P(z) lie in

$$\left|z-\frac{(1-k_1)\sim_n+i(1-k_2)\;_n}{a_n}\right|\le\frac{1}{|a_n|}[\sim_{\text{ª}}+\;_{\text{«}}-(k_1\sim_n+k_2\;_n)+L+M$$

$$+(1-k_1)\sum_{j=\text{ª}+1}^{n-1}(\sim_j+|\sim_j|)+(1-k_2)\sum_{j=\text{«}+1}^{n-1}(\;_j+|\;_j|)].$$

Further the number of zeros of P(z) in $\frac{|a_0|}{X}\le|z|\le\frac{R}{c},c>1,R\ge1$ does not exceed $\frac{1}{\log c}\log\frac{X+|a_0|}{|a_0|}=\frac{1}{\log c}\log(1+\frac{X}{|a_0|})$ and the number of zeros of P(z) in $\frac{|a_0|}{Y}\le|z|\le\frac{R}{c},c>1,R\le1$ does not exceed $\frac{1}{\log c}\log\frac{Y+|a_0|}{|a_0|}=\frac{1}{\log c}\log(1+\frac{Y}{|a_0|})$, where R is any positive number and

$$X=|a_n|R^{n+1}+R^n[\sim_{\text{ª}}+\;_{\text{«}}-(\sim_n+\;_n)+(1-k_1)\sum_{j=\text{ª}+1}^{n}(|\sim_j|+\sim_j)$$

$$+(1-k_2)\sum_{j=\text{«}+1}^{n}(|\;_j|+\;_j)],\ R\ge1$$

$$Y=|a_n|R^{n+1}+R[\sim_{\text{ª}}+\;_{\text{«}}-(\sim_n+\;_n)+(1-k_1)\sum_{j=\text{ª}+1}^{n}(|\sim_j|+\sim_j)$$

$$+(1-k_2)\sum_{j=\text{«}+1}^{n}(|\;_j|+\;_j)],R\le1.$$

For different values of the parameters in Theorem 1, we get different interesting results. For example, for $k_1=k_2=1$, Theorem 1 gives the following result:

Corollary 1: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\sim_j,\mathrm{Im}(a_j)=\;_j,\ j=0,1,2,......,n$ such that for some $\text{ª},\text{«},0\le\text{ª}\le n-1,0\le\text{«}\le n-1$,

$$\alpha_n \le \alpha_{n-1} \le ...... \le \alpha_{\lambda+1} \le \alpha_\lambda ,$$
$$\beta_n \le \beta_{n-1} \le ...... \le \beta_\mu \le \beta_\mu ,$$

and

$$L = |\alpha_\lambda - \alpha_{\lambda-1}| + |\alpha_{\lambda-1} - \alpha_{\lambda-2}| + ... + |\alpha_1 - \alpha_0| + |\alpha_0| ,$$
$$M = |\beta_\mu - \beta_{\mu-1}| + |\beta_{\mu-1} - \beta_{\mu-2}| + ... + |\beta_1 - \beta_0| + |\beta_0| .$$

Then all the zeros of P(z) lie in

$$|z| \le \frac{1}{|a_n|}[\alpha_\lambda + \beta_\mu - (\alpha_n + \beta_n) + L + M] .$$

Further the number of zeros of P(z) in $\frac{|a_0|}{X} \le |z| \le \frac{R}{c}, c > 1, R \ge 1$ does not exceed $\frac{1}{\log c}\log\frac{X + |a_0|}{|a_0|} = \frac{1}{\log c}\log(1 + \frac{X}{|a_0|})$ and the number of zeros of P(z) in $\frac{|a_0|}{Y} \le |z| \le \frac{R}{c}, c > 1, R \le 1$ does not exceed $\frac{1}{\log c}\log\frac{Y + |a_0|}{|a_0|} = \frac{1}{\log c}\log(1 + \frac{Y}{|a_0|})$, where R is any positive number and

$$X = |a_n|R^{n+1} + R^n[\alpha_\lambda + \beta_\mu - (\alpha_n + \beta_n) + L + M] \quad , R \ge 1$$
$$Y = |a_n|R^{n+1} + R[\alpha_\lambda + \beta_\mu - (\alpha_n + \beta_n) + L + M] \quad , R \le 1 .$$

Taking $\lambda = \mu = 0$ in Theorem 1, we get the following result:

Corollary 2: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j , \text{Im}(a_j) = \beta_j , j = 0,1,2,......, n$ such that for some $0 < k_1, k_2 \le 1$,

$$k_1^n \alpha_n \le k_1^{n-1}\alpha_{n-1} \le ...... \le k_1 \alpha_1 \le \alpha_0 ,$$
$$k_2^n \beta_n \le k_2^{n-1}\beta_{n-1} \le ...... \le k_2 \beta_\mu \le \beta_0 .$$

Then all the zeros of P(z) lie in

$$\left| z - \frac{(1-k_1)\alpha_n + i(1-k_2)\beta_n}{a_n} \right| \le \frac{1}{|a_n|}[\alpha_0 + \beta_0 - (k_1\alpha_n + k_2\beta_n) + |\alpha_0| + |\beta_0|] .$$

Further the number of zeros of P(z) in $\frac{|a_0|}{X} \le |z| \le \frac{R}{c}, c > 1, R \ge 1$ does not exceed $\frac{1}{\log c}\log\frac{X + |a_0|}{|a_0|} = \frac{1}{\log c}\log(1 + \frac{X}{|a_0|})$ and the number of zeros of P(z) in $\frac{|a_0|}{Y} \le |z| \le \frac{R}{c}, c > 1, R \le 1$ does not exceed $\frac{1}{\log c}\log\frac{Y + |a_0|}{|a_0|} = \frac{1}{\log c}\log(1 + \frac{Y}{|a_0|})$, where R is any positive number and

$$X = |a_n|R^{n+1} + R^n[\alpha_0 + \beta_0 - (\alpha_n + \beta_n) + |\alpha_0| + |\beta_0|] \quad , R \ge 1$$
$$Y = |a_n|R^{n+1} + R[\alpha_0 + \beta_0 - (\alpha_n + \beta_n) + |\alpha_0| + |\beta_0| \quad , R \le 1 .$$

Lemmas

For the proofs of the above result, we need the following lemmas:

Lemma 1: Let f(z) (not identically zero) be analytic for $|z| \le R, f(0) \ne 0$ and $f(a_k) = 0, \quad k = 1,2,......,n$. Then

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|f(Re^{i\theta})\right|d\theta-\log\left|f(0)\right|=\sum_{j=1}^{n}\log\frac{R}{\left|a_j\right|}.$$

Lemma 2 is the famous Jensen's Theorem (see page 208 of [1]).

Lemma 2: Let f (z) be analytic for $|z|\le R,\ f(0)\neq 0$ and $|f(z)|\le M$ for $|z|\le R$. Then the number of zeros of f(z) in $|z|\le\frac{R}{c}, c>1$ does not exceed $\frac{1}{\log c}\log\frac{M}{|f(0)|}$.

Lemma 2 is a simple deduction from Lemma 1.

Proof of Theorem 1

Consider the polynomial

$$\begin{aligned}
F(z)&=(1-z)P(z)\\
&=(1-z)(a_nz^n+a_{n-1}z^{n-1}+......+a_1z+a_0)\\
&=-a_nz^{n+1}+(a_n-a_{n-1})z^n+......+(a_{\lambda+1}-a_\lambda)z^{\lambda+1}+(a_\lambda-a_{\lambda-1})z^\lambda\\
&\quad+......+(a_1-a_0)z+a_0\\
&=-a_nz^{n+1}-(k_1-1)\beta_nz^n+(k_1\beta_n-\beta_{n-1})z^n+(k_1\beta_{n-1}-\beta_{n-2})z^{n-1}+......\\
&\quad+(k_1\beta_{\lambda+1}-\beta_\lambda)z^{\lambda+1}+(\beta_\lambda-\beta_{\lambda-1})z^\lambda+......+(\beta_1-\beta_0)z+\beta_0\\
&\quad-(k_1-1)(\beta_{n-1}z^{n-1}+\beta_{n-2}z^{n-2}+.......+\beta_{\lambda+1}z^{\lambda+1})\\
&\quad+i\{(k_2\alpha_n-\alpha_{n-1})z^n-(k_2-1)\alpha_nz^n+(k_2\alpha_{n-1}-\alpha_{n-2})z^{n-1}+......\\
&\quad+(k_2\alpha_{\mu+1}-\alpha_\mu)z^{\mu+1}-(k_2-1)(\alpha_{n-1}z^{n-1}+......+\alpha_{\mu+1}z^{\mu+1})+(\alpha_\mu-\alpha_{\mu-1})z^\mu\\
&\quad+......+(\alpha_1-\alpha_0)z+\alpha_0\}
\end{aligned}$$

For $|z|>1$ so that $\frac{1}{|z|^j}<1,\forall j=1,2,......,n$, we have, by using the hypothesis

$$\begin{aligned}
|F(z)|&\ge\left|a_nz+(k_1-1)\beta_n+i(k_2-1)\alpha_n\right||z|^n-[\left|k_1\beta_n-\beta_{n-1}\right||z|^n+\left|k_1\beta_{n-1}-\beta_{n-2}\right||z|^{n-1}+......\\
&\quad+\left|k_1\beta_{\lambda+1}-\beta_\lambda\right||z|^{\lambda+1}+\left|\beta_\lambda-\beta_{\lambda-1}\right||z|^\lambda+......+\left|\beta_1-\beta_0\right||z|+\left|\beta_0\right|\\
&\quad+(1-k_1)(\left|\beta_{n-1}\right||z|^{n-1}+......+\left|\beta_{\lambda+1}\right||z|^{\lambda+1})+\left|k_2\alpha_n-\alpha_{n-1}\right||z|^n+\left|k_2\alpha_{n-1}-\alpha_{n-2}\right||z|^{n-1}\\
&\quad+......+\left|k_2\alpha_{\mu+1}-\alpha_\mu\right||z|^{\mu+1}+\left|\alpha_\mu-\alpha_{\mu-1}\right||z|^\mu+......+\left|\alpha_1-\alpha_0\right||z|+\left|\alpha_0\right|\\
&\quad+(1-k_2)(\left|\alpha_{n-1}\right||z|^{n-1}+......+\left|\alpha_{\mu+1}\right||z|^{\mu+1})]\\
&=|z|^n[\left|a_nz+(k_1-1)\beta_n+i(k_2-1)\alpha_n\right|-[\left|k_1\beta_n-\beta_{n-1}\right|+\frac{\left|k_1\beta_{n-1}-\beta_{n-2}\right|}{|z|}+\frac{\left|k_1\beta_{n-2}-\beta_{n-3}\right|}{|z|^2}\\
&\quad+......+\frac{\left|k_1\beta_{\lambda+1}-\beta_\lambda\right|}{|z|^{n-\lambda-1}}+\frac{\left|\beta_\lambda-\beta_{\lambda-1}\right|}{|z|^{n-\lambda}}+......+\frac{\left|\beta_1-\beta_0\right|}{|z|^{n-1}}+\frac{\left|\beta_0\right|}{|z|^n}+(k_1-1)(\frac{\left|\beta_{n-1}\right|}{|z|}+......\\
&\quad+\frac{\left|\beta_{\lambda+1}\right|}{|z|^{n-\lambda-1}})+\left|k_2\alpha_n-\alpha_{n-1}\right|+\frac{\left|k_2\alpha_{n-1}-\alpha_{n-2}\right|}{|z|}+......+\frac{\left|k_2\alpha_{\mu+1}-\alpha_\mu\right|}{|z|^{n-\mu-1}}+\frac{\left|\alpha_\mu-\alpha_{\mu-1}\right|}{|z|^{n-\mu}}\\
&\quad+......+\frac{\left|\alpha_1-\alpha_0\right|}{|z|^{n-1}}+\frac{\left|\alpha_0\right|}{|z|^n}+(k_2-1)(\frac{\left|\alpha_{n-1}\right|}{|z|}+......+\frac{\left|\alpha_{\mu+1}\right|}{|z|^{n-\mu-1}})]\\
&>|z|^n[\left|a_nz+(k_1-1)\beta_n+i(k_2-1)\alpha_n\right|-\{\left|k_1\beta_n-\beta_{n-1}\right|+\left|k_1\beta_{n-1}-\beta_{n-2}\right|+\left|k_1\beta_{n-2}-\beta_{n-3}\right|
\end{aligned}$$

$$+ \ldots\ldots + |k_1\alpha_{\lambda+1} - \alpha_\lambda| + |\alpha_\lambda - \alpha_{\lambda-1}| + \ldots\ldots + |\alpha_1 - \alpha_0| + |\alpha_0| + (1-k_1)(|\alpha_{n-1}| + \ldots\ldots$$
$$+ |\alpha_{\lambda+1}|) + |k_2\beta_n - \beta_{n-1}| + |k_2\beta_{n-1} - \beta_{n-2}| + \ldots\ldots + |k_2\beta_{\mu+1} - \beta_\mu| + |\beta_\mu - \beta_{\mu-1}|$$
$$+ \ldots\ldots + |\beta_1 - \beta_0| + |\beta_0| + (1-k_2)(|\beta_{n-1}| + \ldots\ldots + |\beta_{\mu+1}|)\}]$$
$$= |z|^n[|a_n z + (k_1-1)\alpha_n + i(k_2-1)\beta_n| - \{\alpha_{n-1} - k_1\alpha_n + \alpha_{n-2} - k_1\alpha_{n-1} + \alpha_{n-3} - k_1\alpha_{n-2}$$
$$+ \ldots\ldots + \alpha_\lambda - k_1\alpha_{\lambda+1} + L + (1-k_1)(|\alpha_{n-1}| + \ldots\ldots + |\alpha_{\lambda+1}|) + \beta_{n-1} - k_2\beta_n$$
$$+ \beta_{n-2} - k_2\beta_{n-1} + \ldots\ldots + \beta_\mu - k_2\beta_{\mu+1} + M + (1-k_2)(|\beta_{n-1}| + \ldots\ldots + |\beta_{\mu+1}|)\}]$$
$$= |z|^n[|a_n z - \{(1-k_1)\alpha_n + i(1-k_2)\beta_n\}| - \{\alpha_\lambda + \beta_\mu - (k_1\alpha_n + k_2\beta_n) + L + M$$
$$+ (1-k_1)\sum_{j=\lambda+1}^{n-1}(\alpha_j + |\alpha_j|) + (1-k_2)\sum_{j=\mu+1}^{n-1}(\beta_j + |\beta_j|)\}]$$
$$> 0$$

if

$$|a_n z - \{(1-k_1)\alpha_n + i(1-k_2)\beta_n\}| > \alpha_\lambda + \beta_\mu - (k_1\alpha_n + k_2\beta_n) + L + M$$
$$+ (1-k_1)\sum_{j=\lambda+1}^{n-1}(\alpha_j + |\alpha_j|) + (1-k_2)\sum_{j=\mu+1}^{n-1}(\beta_j + |\beta_j|)$$

i.e. if

$$\left|z - \frac{(1-k_1)\alpha_n + i(1-k_2)\beta_n}{a_n}\right| > \frac{1}{|a_n|}[\alpha_\lambda + \beta_\mu - (k_1\alpha_n + k_2\beta_n) + L + M$$
$$+ (1-k_1)\sum_{j=\lambda+1}^{n-1}(\alpha_j + |\alpha_j|) + (1-k_2)\sum_{j=\mu+1}^{n-1}(\beta_j + |\beta_j|)].$$

This shows that those zeros of F(z) whose modulus is greater than 1 lie in

$$\left|z - \frac{(1-k_1)\alpha_n + i(1-k_2)\beta_n}{a_n}\right| \le \frac{1}{|a_n|}[\alpha_\lambda + \beta_\mu - (k_1\alpha_n + k_2\beta_n) + L + M$$
$$+ (1-k_1)\sum_{j=\lambda+1}^{n-1}(\alpha_j + |\alpha_j|) + (1-k_2)\sum_{j=\mu+1}^{n-1}(\beta_j + |\beta_j|)].$$

Since the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality and since the zeros of P(z) are also the zeros of F(z) , it follows that all the zeros of P(z) lie in

$$\left|z - \frac{(1-k_1)\alpha_n + i(1-k_2)\beta_n}{a_n}\right| \le \frac{1}{|a_n|}[\alpha_\lambda + \beta_\mu - (k_1\alpha_n + k_2\beta_n) + L + M$$
$$+ (1-k_1)\sum_{j=\lambda+1}^{n-1}(\alpha_j + |\alpha_j|) + (1-k_2)\sum_{j=\mu+1}^{n-1}(\beta_j + |\beta_j|)].$$

Again

$F(z) = a_0 + G(z)$,

where

$$G(z) = -a_n z^{n+1} - (k_1-1)\alpha_n z^n + (k_1\alpha_n - \alpha_{n-1})z^n + (k_1\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \ldots\ldots$$
$$+ (k_1\alpha_{\lambda+1} - \alpha_\lambda)z^{\lambda+1} + (\alpha_\lambda - \alpha_{\lambda-1})z^\lambda + \ldots\ldots + (\alpha_1 - \alpha_0)z$$
$$- (k_1-1)(\alpha_{n-1}z^{n-1} + \alpha_{n-2}z^{n-2} + \ldots\ldots. + \alpha_{\lambda+1}z^{\lambda+1})$$
$$+ i\{(k_2\beta_n - \beta_{n-1})z^n - (k_2-1)\beta_n z^n + (k_2\beta_{n-1} - \beta_{n-2})z^{n-1} + \ldots\ldots$$
$$+ (k_2\beta_{\mu+1} - \beta_\mu)z^{\mu+1} - (k_2-1)(\beta_{n-1}z^{n-1} + \ldots\ldots + \beta_{\mu+1}z^{\mu+1}) + (\beta_\mu - \beta_{\mu-1})z^\mu$$
$$+ \ldots\ldots + (\beta_1 - \beta_0)z\}.$$

For $|z| = R, R > 0$, we have

$$|G(z)| \le |a_n|R^{n+1} + (1-k_1)|\sim_n|R^n + (\sim_{n-1} - k_1\sim_n)R^n + (\sim_{n-2} - k_1\sim_{n-1})R^{n-1} + ...$$

$$+ (\sim_{ª} - k_1\sim_{ª+1})R^{ª+1} + |\sim_{ª} - \sim_{ª-1}|R^{ª} + ... + |\sim_1 - \sim_0|R$$

$$+ (1-k_1)(|\sim_{n-1}|R^{n-1} + |\sim_{n-2}|R^{n-2} + ... + |\sim_{ª+1}|R^{ª+1})$$

$$+ (\quad _{n-1} - k_2 \quad _n)R^n + (1-k_2)| \quad _n|R^n + (\quad _{n-2} - k_2 \quad _{n-1})R^{n-1} + ...$$

$$+ (\quad _{«} - k_2 \quad _{«+1})R^{«+1} + (1-k_2)(| \quad _{n-1}|R^{n-1} + ... + | \quad _{«+1}|R^{«+1}) + | \quad _{«} - \quad _{«-1}|R^{«}$$

$$+ ...... + | \quad _1 - \quad _0|R\}$$

$$\le |a_n|R^{n+1} + R^n[(1-k_1)|\sim_n| + \sim_{n-1} - k_1\sim_n + \sim_{n-2} - k_1\sim_{n-1} + ... + \sim_{ª} - k_1\sim_{ª+1} + L$$

$$+ (1-k_1)(|\sim_{n-1}| + ... + |\sim_{ª+1}|) + \quad _{n-1} - k_2 \quad _n$$

$$+ \quad _{n-2} - k_2 \quad _{n-1} + ... + \quad _{«} - k_2 \quad _{«+1} + M + (1-k_2)(| \quad _{n-1}| + ... + | \quad _{«+1}|)]$$

$$= |a_n|R^{n+1} + R^n[\sim_{ª} + \quad _{«} - (\sim_n + \quad _n) + (1-k_1)\sum_{j=ª+1}^{n}(|\sim_j| + \sim_j)$$

$$+ (1-k_2)\sum_{j=«+1}^{n}(| \quad _j| + \quad _j)]$$

$$= X$$

for $R \ge 1$ and
for $R \le 1$

$$|G(z)| \le |a_n|R^{n+1} + R[\sim_{ª} + \quad _{«} - (\sim_n + \quad _n) + (1-k_1)\sum_{j=ª+1}^{n}(|\sim_j| + \sim_j)$$

$$+ (1-k_2)\sum_{j=«+1}^{n}(| \quad _j| + \quad _j)]$$

$$= Y$$

Since G(0)=0 and G(z) is analytic for $|z| \le R$, it follows, by Schwarz Lemma , that for $|z| \le R$,

$|G(z)| \le X|z|$ for $R \ge 1$ and $|G(z)| \le Y|z|$ for $R \le 1$.

Hence, for $|z| \le R$, $R \ge 1$

$$|F(z)| = |a_0 + G(z)|$$

$$\ge |a_0| - |G(z)|$$

$$\ge |a_0| - X|z|$$

$$> 0$$

if

$$|z| < \frac{|a_0|}{X} .$$

Similarly, for $|z| \le R$, $R \le 1$, $|F(z)| > 0$ if $|z| < \frac{|a_0|}{Y}$.

In other words, F(z) does not vanish in $|z| < \frac{|a_0|}{X}$ for $R \geq 1$ and F(z) does not vanish in $|z| < \frac{|a_0|}{Y}$ for $R \leq 1$ in $|z| \leq R$. That means all the zeros of F(z) and hence all the zeros of P(z) lie in $|z| \geq \frac{|a_0|}{X}$ for $R \geq 1$ and in $|z| \geq \frac{|a_0|}{Y}$ that for $R \leq 1$ in $|z| \leq R$.

Since for $|z| \leq R$, $|F(z)| \leq X + |a_0|$ for $R \geq 1$ and $|F(z)| \leq Y + |a_0|$ for $R \leq 1$ and since

$F(0) = a_0 \neq 0$, it follows by Lemma 2 that the number of zeros of P(z) in $\frac{|a_0|}{X} \leq |z| \leq \frac{R}{c}, c > 1, R \geq 1$ does not exceed $\frac{1}{\log c}\log\frac{X + |a_0|}{|a_0|} = \frac{1}{\log c}\log(1 + \frac{X}{|a_0|})$ and the number of zeros of P(z) in $\frac{|a_0|}{Y} \leq |z| \leq \frac{R}{c}, c > 1, R \leq 1$ does not exceed $\frac{1}{\log c}\log\frac{Y + |a_0|}{|a_0|} = \frac{1}{\log c}\log(1 + \frac{Y}{|a_0|})$.

That proves Theorem 1 completely.

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Please cite this article in press as:
Gulzar M.H (2017), On the Location of Zeros of Polynomials, *International Journal of Current Advanced Research*, 6(3), pp. 2351-2357
http://dx.doi.org/10.24327/ijcar.2017. 2357.0008

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