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Research Article

ON FINDING INTEGER SOLUTIONS TO THE TERNARY QUADRATIC DIOPHANTINE EQUATION $2(x^2 + y^2) - 3xy = 43z^2$

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A B S T R A C T

The homogeneous ternary quadratic equation given by $2(x^2 + y^2) - 3xy = 43z^2$ is analysed for its non-zero distinct integer solutions through different methods. Also, formulae for generating sequence of integer solutions based on the given solution are presented.

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INTRODUCTION

Ternary quadratic equations are rich in variety [1- 4, 17-19]. For an extensive review of sizable literature and various problems, one may refer [5-16]. In this communication, yet another interesting homogeneous ternary quadratic Diophantine equation given by $2(x^2 + y^2) - 3xy = 43z^2$ is analysed for its non-zero distinct integer solutions through different methods. Also, formulas for generating sequence of integer solutions based on the given solution are presented.

Method of analysis

The ternary quadratic Diophantine equation to be solved for non-zero distinct integral solution is

$$2(x^2 + y^2) - 3xy = 43z^2 \quad (1)$$

Introduction of the linear transformations

$$x = u + v \ , y = u - v \ , u \neq v \neq 0 \quad (2)$$

in (1) leads to

$$u^2 + 7v^2 = 43z^2 \quad (3)$$

The above equation is solved for u, v and z through different ways and using (2), the values of x and y satisfying (1), are obtained which are illustrated below:

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Assume

$$z = a^2 + 7b^2 \quad (4)$$

Write 43 as

$$43 = (6 + i\sqrt{7})(6 - i\sqrt{7}) \quad (5)$$

Using (4) and (5) in (3) and employing the method of factorization, define

$$(u + i\sqrt{7}v) = (6 + i\sqrt{7})(a + i\sqrt{7}\,b)^2$$

Equating the real and imaginary parts, we get

$$u = 6a^2 - 14ab - 42b^2$$

$$v = a^2 + 12ab - 7b^2$$

In view of (2), one obtains

$$\left.\begin{aligned} x &= 7a^2 - 2ab - 49b^2 \\ y &= 5a^2 - 26ab - 35b^2 \end{aligned}\right\} \quad (6)$$

Thus (4) and (6) represent the integer solution to (1).

Way2:

One can write 43 as

$$43 = \frac{(25 + i3\sqrt{7})(25 - i3\sqrt{7})}{4^2} \quad (7)$$

Using (4) and (7) in (3) and applying the method of factorization, define

$$(u+i\sqrt{7}v)=\frac{(25+i3\sqrt{7})(a+i\sqrt{7}b)^2}{4}$$

Equating the real and imaginary parts, we get

$$u=\frac{25a^2-42ab-175b^2}{4}$$

$$v=\frac{3a^2+50ab-21b^2}{4}$$

In view of (2), one obtains

$$\left.\begin{aligned} x&=\frac{28a^2+8ab-196b^2}{4}\\ y&=\frac{22a^2-92ab-154b^2}{4}\end{aligned}\right\} \quad (8)$$

To obtain the integer solutions, replacing a by 2A and b by 2B in (4) & (8) , the corresponding integral solutions of (1) are given by

$$\left.\begin{aligned} x&=28A^2+8AB-196B^2\\ y&=22A^2-92AB-154B^2\\ z&=4A^2+28B^2\end{aligned}\right\} \quad (9)$$

Way3:

(3) can be written as

$$u^2+7v^2=43z^2*1 \quad (10)$$

Write 1 on the R.H.S. of (10) as

$$1=\frac{(3+i\sqrt{7})(3-i\sqrt{7})}{4^2} \quad (11)$$

Using (4), (5) and (11) in (10) and utilizing the method of factorization, define

$$(u+i\sqrt{7}v)=(6+i\sqrt{7})(a+i\sqrt{7}b)^2\left[\frac{(3+i\sqrt{7})}{4}\right]$$

Equating the real and imaginary parts, the values of u and v are obtained as

$$u=\frac{11a^2-126ab-77b^2}{4}$$

$$v=\frac{9a^2+22ab-63b^2}{4}$$

Proceeding as in Way2, we get

$$\left.\begin{aligned} x&=20A^2-104AB-140B^2\\ y&=2A^2-148AB-14B^2\\ z&=4A^2+28B^2\end{aligned}\right\} \quad (12)$$

Thus (12) represent the non-zero distinct solution of (1)

Way 4:

We can write 1 on the R.H.S. of (10) as

$$1=\frac{(1+3i\sqrt{7})(1-3i\sqrt{7})}{8^2} \quad (13)$$

Using (4),(5) and (13) in (10) and by factorization method define

$$(u+i\sqrt{7}v)=\frac{(6+i\sqrt{7})(a+i\sqrt{7}b)^2\,(1+3i\sqrt{7})}{8}$$

Equating the real and imaginary parts, we get

$$u=\frac{-15a^2-266ab+105b^2}{8}$$

$$v=\frac{19a^2-30ab-133b^2}{8}$$

By proceeding as in Way2, we obtain

$$x=2A^2-148AB-14B^2$$

$$y=-17A^2-118AB+119B^2$$

$$z=4A^2+28B^2$$

which represents the non-zero integral solution of (1).

Way 5:

Write (3) in the form of ratio as

$$\frac{u+6z}{7(z-v)}=\frac{z+v}{u-6z}=\frac{a}{b},\ b\neq 0,$$

which is equivalent to the system of double equations

$$bu+7av+(6b-7a)z=0$$

$$au-bv+(-6a-b)z=0$$

Solving the above system of double equations and using (2),the corresponding integer solutions to (1) are found to be

$$x=49a^2+2ab-7b^2$$

$$y=35a^2+26ab-5b^2$$

$$z=7a^2+b^2$$

Note1:

It is noted that (3) may also be written in the form of ratios as below:

$$(i)\quad \frac{u+6z}{z+v}=\frac{7(z-v)}{u-6z}=\frac{a}{b}$$

$$(ii)\quad \frac{u-6z}{7(z-v)}=\frac{z+v}{u+6z}=\frac{a}{b}$$

$$(iii)\quad \frac{u-6z}{z+v}=\frac{7(z-v)}{u+6z}=\frac{a}{b}$$

$$(iv)\quad \frac{u+6z}{7(z+v)}=\frac{(z-v)}{u-6z}=\frac{a}{b}$$

$$(v)\quad \frac{u-6z}{z-v}=\frac{7(z+v)}{u+6z}=\frac{a}{b}$$

For each of the above ratios, the corresponding integer solutions to (1) are exhibited below:

Solutions obtained through (i)

$$x=5a^2+26ab-35b^2$$
$$y=7a^2+2ab-49b^2$$
$$z=a^2+7b^2$$

Solutions obtained through (ii)

$$x=35a^2-26ab-5b^2$$
$$y=49a^2-2ab-7b^2$$
$$z=-7a^2-b^2$$

Solutions obtained through (iii)

$$x=-7a^2+2ab+49b^2$$
$$y=-5a^2+26ab+35b^2$$
$$z=a^2+7b^2$$

Solutions obtained through (iv)

$$x=35a^2+26ab-5b^2$$
$$y=49a^2+2ab-7b^2$$
$$z=7a^2+b^2$$

Solutions obtained through (v)

$$x=5a^2-26ab-35b^2$$
$$y=7a^2-2ab-49b^2$$
$$z=a^2+7b^2$$

Way6

Introducing the linear transformations

$$z=X+7R\ ,v=X+43R\ ,u=6U \quad (14)$$

in (3),it gives

$$X^2=301R^2+U^2 \quad (15)$$

which is satisfied by

$$X=r^2+301s^2, U=r^2-301s^2, R=2rs \quad (16)$$

From (16), (14) &(2), we obtain the integer solutions to (1) as given below:

$$x=7r^2+86rs-1505s^2$$
$$y=5r^2-86rs-2107s^2$$
$$z=r^2+301s^2+14rs$$

It is to be noted that (15) may be represented as the system of double equation as shown in **Table: 1**

Table 1 System of double equations

System	1	2	3	4	5	6	7	8	9	10	11	12
$X+U$	$301R^2$	$43R^2$	$7R^2$	R^2	301	43	7	1	$301R$	$43R$	$7R$	R
$X-U$	1	7	43	301	R^2	$7R^2$	$43R^2$	$301R^2$	R	$7R$	$43R$	$301R$

Solving each of the system of double equations in Table:1,the values of X,U and R are obtained. From (14) & (2),the corresponding solutions to (1) are found and They are exhibited below:

Solutions from system1

$$x=4214k^2+4300k+1094$$
$$y=3010k^2+2924k+706$$
$$z=602k^2+616k+158$$

Solutions from system2

$$x=602k^2+688k+176$$
$$y=430k^2+344k+40$$
$$z=86k^2+100k+32$$

Solutions from system3

$$x=98k^2+184k-40$$
$$y=70k^2-16k-176$$
$$z=14k^2+28k+32$$

Solutions from system4

$$x=14k^2+100k-706$$
$$y=10k^2-76k-1094$$
$$z=2k^2+16k+158$$

Solutions from system5

$$x=-10k^2+76k+1094$$
$$y=-14k^2-100k+706$$
$$z=2k^2+16k+158$$

Solutions from system6

$$x=-70k^2+16k+176$$
$$y=-98k^2-184k+40$$
$$z=14k^2+28k+32$$

Solutions from system7

$$x=-430k^2-344k-40$$
$$y=-602k^2-688k-176$$
$$z=86k^2+100k+32$$

Solutions from system8

$$x = -3010k^2 - 2924k - 706$$

$$y = -4214k^2 - 4300k - 1094$$

$$z = 602k^2 + 616k + 158$$

Solutions from system9

$$x = 1094R$$

$$y = 706R$$

$$z = 158R$$

Solutions from system10

$$x = 176R$$

$$y = 40R$$

$$z = 32R$$

Solutions from system11

$$x = -40R$$

$$y = -176R$$

$$z = 32R$$

Solutions from system12

$$x = -706R$$

$$y = -1094R$$

$$z = 158R$$

CONCLUSION

To conclude, one may search for other patterns of solutions and their corresponding properties.

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