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Research Article

THE DYNAMICS OF PREY-PREDATOR MODEL WITH HARVESTING INVOLVING DISEASES IN BOTH POPULATIONS

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A B S T R A C T

In this paper, a mathematical model consisting of the prey-predator model with different infectious diseases that spreads in both population and harvesting in the infected population. It is assumed that the disease is not transmitted from prey to predator or conversely, in addition to that both diseases spread within same species by contact between susceptible and infected individuals. Two types of functional response for describing the predation as well as linear incidence for describing transition of diseases are used. The existence, uniqueness, boundedness of the solution and the stability analysis of all possible equilibrium points are studied. The Lyapunov function is used to study the global dynamics of the model. The effect of the disease and harvest on the dynamical of the system is discussed by using numerical simulation.

INTRODUCTION

A mathematical model is an abstract representation of a real-world phenomenon that uses mathematical language to describe the behavior of a natural or physical system. Typically mathematical formula is used to describe the interactions of the various components of the system by depending on two important different fields; these are the ecology and epidemiology. The ecology is the branch of biology that deals with the relations and interactions between organisms and their environment, while the epidemiology is the branch of medicine that deals with the incidence and prevalence of disease in large populations and with detection of the source and cause of epidemics of infectious disease. These fields are studied extensively in literatures for long time as separated fields. Anderson and May [1] were the first whose merged the above two fields and formulated Lotka-Volterra predator-prey model with infection disease spread among prey by contact between them and no reproduction in infected prey.

In fact, the study of effect of infectious disease in ecological system is now becoming an important factor for regulating animal and human population size. So in the last years; mathematical models have become extremely important tools in analyzing and understanding the spread and control of infectious disease through the study of the different types from disease for example SI, SIS and SIR. Where some of infectious disease in the ecology system is transmitted by contact in same of species have proposed and studied from some of researchers, Naji and Mustafa [4] studied a prey-predator model with SI infectious disease in prey, while Ramana Murthy and Bahlool [26] studied a prey-predator model with SI infectious disease in predator. Moreover, there are some of infectious diseases are transmitted in the species not only through contact, but also directly from environment. Majeed and Shawka [2] studied prey-predator model with SI and SIS infectious disease in prey population and the disease transmitted within the same species by contact and external source. In addition to Khalaf, Majeed and Naji [3] studied prey-predator model with SIS infectious disease in prey population this disease passed from a prey to predator through attacking of predator to prey and the disease transmitted within the same species by contact and external source.

The harvest rate has a strong influence on the dynamic development of the population, perhaps one of the most important hunting the fish or eradication on the disease. Bhattacharyya and Mukhopadhyay [5] studied prey-predator model with harvest and disease, and he assumed that the harvest can eradication the disease, also Bairagi el.at [6] studied prey-predator model with harvest and disease ,and he assumed that the harvest can remove a parasite , In general, there are three kinds of harvesting function [7, 8, 9] have been studied in the literature

1. Constant harvesting

$K(x,V) = C,$

where $k(x,V)$ is the harvesting, C is a acceptable constant

2. Proportionate harvesting

$K(x, V) = qVx,$

where q is the catchability of the species. V is the harvesting effect.

3. Nonlinear harvesting

$K(x, V) = \frac{qVx}{b_1V+b_2x},$

where b_1, b_2 are acceptable positive constants.

Some studies that address the population contain the harvest, Brauer and Soudack [10,11] studied a predator-prey model under constant rate of harvesting. On other hand there are many studies includes disease and proportionate harvesting , Abd ul Satar[12] studied a prey-predator model with disease SIS-type and harvesting on the prey and the predator , while Sujatha and Gunasekaran [13], Wuhaih[14] and Agnihotri[15] studied a prey-predator model with disease SIS-type, SI-type and harvesting in prey only, in addition so many researchers have predator-prey systems that contain nonlinear harvesting functions [16- 19], while Some of the studies using time delay with harvest were considered by Aiello and Freedman [20], Rosen [21] ,Freedman and Gopaisammy [22], Cushing and Saleem [23].

Recently, Bera et al. [24] had proposed and studied a prey-predator model involving, SI infectious diseases in prey and predator species; in addition to the disease is not transmitted from a prey to predator or conversely. It is assumed that both the diseases spread within prey and predator population by contact, between susceptible individuals and infected individuals. Furthermore, he used linear functional response and linear incidence rate to describe spread both diseases.

In this section, an eco-epidemiological mathematical model consisting of prey-predator model involving SI infectious diseases in prey and predator species with harvesting in infectious population has been proposed and analyzed. Further, in this model, Holling type-II functional response for the predation of susceptible prey and linear functional response for the predation of infected prey as well as linear incidence rate for describing the transition of disease are used. Our aim is to study the effect of harvesting on the dynamics of disease propagation and eradication it.

MATHEMATICAL MODEL

In this section, an eco-epidemiological model is proposed for study. The model consists of a prey, whose total population density at time T is denoted by $N(T)$, interacting with predator whose total population at time T id denoted by $P(T)$. It is assumed that both the prey and the predator populations are infected by different infectious disease. Now, the following assumptions are adopted in formulating the basic eco-epidemiology model:

1. There is an SI epidemic disease in both prey and predator population's divides the prey population into two classes namely $S(T)$ that represents the density of susceptible prey at time T and $I(T)$ which represents the density of infected prey at time$T$. Therefore at any time T, we have $N(T) = S(T) + I(T)$. Also divides the predator population in to two classes namely $X(T)$ that represents the density of susceptible predator at time T and $Y(T)$ which represents the density of infected predator at time$T$. Therefore at any time T, we have $P(T) = X(T) + Y(T)$.
2. It is assumed that only susceptible prey S is capable of reproducing in logistic growth with carrying capacity K>0 and intrinsic growth rate constant r>0, the infected prey I is removed before having the possibility of reproducing. However, the infected prey population I still contribute with S to population growth toward the carrying capacity.
3. The disease is transmitted within the same species by contact with an infected individual at infection rates $\beta_1 > 0$ and $\beta_2 > 0$ for the prey and predator respectively.
4. The susceptible predator consumes the susceptible and infected prey according to Holling type-II and Lotka-Volterra of functional response with maximum attack rate $a_1 > 0$ and half saturation rate $b > 0$ for susceptible prey and maximum attack rate $a_2 > 0$ for infected prey respectively, while the infected predator consume the infected prey according to Lotka-Volterra of functional response with maximum attack rate $a_3 > 0$, and contribute a portion of such food with conversion rates $e_i > 0; i = 1,2,3$.
5. In the absence of the prey the susceptible and infected predator decay exponentially with natural death rate $d_2 > 0$.
6. The disease may causes mortality with a constant mortality rates $d_1 > 0$ and $\alpha > 0$ for the infected prey and infected predator respectively.
7. Finally, the infected populations are harvest with constant rates $h_1 > 0$ and $h_2 > 0$ for the prey and predator respectively.
8. According to the above assumptions, the proposed mathematical model can be represented mathematically by the following set of first order non-linear differential equations.

$$\begin{aligned} \frac{ds}{dT} &= rs\left(1 - \frac{s+I}{k}\right) - \beta_1 SI - \frac{a_1SX}{b+S} \\ \frac{dI}{dT} &= \beta_1 SI - a_2IX - a_3IY - d_1I - h_1I \\ \frac{dX}{dT} &= e_1\frac{a_1SX}{b+S} + e_2a_2IX - \beta_2XY - d_2X \\ \frac{dY}{dT} &= \beta_2XY + e_3a_3IY - (d_2+\alpha)Y - h_2Y \end{aligned} \tag{2.1}$$

With initial conditions $S(0) \geq 0, I(0) \geq 0, X(0) \geq 0$ and $Y(0) \geq 0$.

Note that the above proposed model has (16) parameters which makes the mathematical analysis of the system difficult. So in order to reduce the number of parameters and determine which parameter represents the control parameter, the following dimensionless variables are used:

$t = r\,T\,, x = \frac{S}{k}\ , y = \frac{I}{k}\ , z = \frac{X}{k}\ , w = \frac{Y}{k}\,.$

Then system (2.1) can be written in the following dimensionless form:

$$\begin{aligned}
\frac{dx}{dt} &= x\left(1 - x - y - c_1 y - \frac{c_2 z}{c_3 + x}\right) = f_1(x, y, z, w)\\
\frac{dy}{dt} &= y\big(c_1 x - c_4 z - c_5 w - (c_6 + c_7)\big) = f_2(x, y, z, w)\\
\frac{dz}{dt} &= z\left(\frac{c_8 x}{c_3 + x} + c_9 y - c_{10} w - c_{11}\right) = f_3(x, y, z, w)\\
\frac{dw}{dt} &= w\big(c_{10} z + c_{12} y - (c_{11} + c_{13} + c_{14})\big) = f_4(x, y, z, w)
\end{aligned} \qquad (2.2)$$

Where

$c_1 = \frac{\beta_1 k}{r}\,, c_2 = \frac{a_1}{r}\,, c_3 = \frac{b}{k}\,, c_4 = \frac{a_2 k}{r}\,, c_5 = \frac{a_3 k}{r}\,, c_6 = \frac{d_1}{r}\,, c_7 = \frac{h_1}{r}\,, c_8 = \frac{e_1 a_1}{r}\,, c_9 = \frac{e_2 a_2 k}{r}\,,$
$c_{10} = \frac{\beta_2 k}{r}\,, c_{11} = \frac{d_2}{r}\,, c_{12} = \frac{e_3 a_3 k}{r}\,, c_{13} = \frac{\alpha}{r}, c_{14} = \frac{h_2}{r}\ .$
With $x(0) \geq 0$,$y(0) \geq 0$, $z(0) \geq 0$ and $w(0) \geq 0$.

represent the dimensionless parameter of system (2.2). It is observed that the number of parameters have been reduced from sixteen in the system (2.1) to fourteen in the system (2.2).

It is easy to verify that all the interaction functions f_1, f_2, f_3 and f_4 on the right hand side of system (2.2) are continuous and have continuous partial derivatives on R_+^4 with respect to dependent variables x , y , z and w. Accordingly they are Lipschitzian functions and hence system (2.2) has a unique solution for each non-negative initial condition. Further the boundedness of the system is shown in the following theorem.

Theorem (2.1): All the solutions of system (2.2) which initiate in R_+^4 are uniformly bounded.

Proof.

Let $(x(t), y(t), z(t), w(t))$ be any solution of the system (2.2) with non-negative initial condition $(x(0), y(0), z(0), w(0))$. According to the first equation of system (2.2) we have:
$$\frac{dx}{dt} \leq x(1 - x)$$
Clearly according to the theory of differential inequality, we get:

$\lim_{t\to\infty}\sup x(t) \leq 1$. Define the function
$M(t) = x(t) + y(t) + z(t) + w(t)$
Therefore
$$\frac{dM}{dt} < 2x - x - \frac{(c_2 - c_8)xz}{c_3 + x} - (c_4 - c_9)zy - (c_5 - c_{12})wy - (c_6 + c_7)y - c_{11} z - (c_{11} + c_{13} + c_{14})w$$

Now, since the conversion rate constant from prey population to predator population can't be exceeding the maximum predation rate constant of predator population to prey population, hence from the biological point of view, always $c_9 < c_4$, $c_{12} < c_5$ and $c_8 < c_2$, hence it is obtained that:

$$\frac{dM}{dt} \leq 2 - nM \qquad \text{where } n = \min\{1, c_6 + c_7, c_{11}, c_{11} + c_{13} + c_{14}\}.$$

Now, by using the comparison theorem [25] on the above differential inequality, we get that:
$$M(t) \leq \frac{2}{n} + \left(M(0) - \frac{2}{n}\right) e^{-nt} \ .$$
Thus $0 \leq M(t) \leq \frac{2}{n}$ as $t \to \infty$. Hence all the solutions of system (2.2) are uniformly bounded and the proof is complete

Existence of equilibrium points

In this section, the existence of all possible equilibrium points of the system (2.2) is discussed. it is observed that , system (2.2) has at most seven equilibrium points.

1. The vanishing equilibrium point $E_0 = (0,0,0,0)$ always exist.
2. The axial equilibrium point $E_1 = (1,0,0,0)$ always exist.
3. The predator-free equilibrium point $E_2 = (\bar{x},\bar{y},0,0)$;

where $\bar{x} = \frac{c_6+c_7}{c_1}$ and $\bar{y} = \frac{c_1-(c_6+c_7)}{c_1(c_1+1)}$,

exists a unique in the *int.* R_+^2 of xy-plane provided that:

$$c_1 > c_6 + c_7 \quad (3.1)$$

4.The disease-free equilibrium point $E_3 = (\dot{x}, 0, \dot{z}, 0)$;

where $\dot{x} = \frac{c_3 c_{11}}{c_8-c_{11}}$ and $\dot{z} = \frac{c_3}{c_2}\left(1-\frac{c_3 c_{11}}{c_8-c_{11}}\right)\left(1+\frac{c_{11}}{c_8-c_{11}}\right)$

exists a unique in the *int.* R_+^2 of xz-plane provided that:

$$c_8 > c_{11} \quad (3.2)$$

$$c_8 > c_{11}(1+c_3) \quad (3.3)$$

5.The infected-predator-free equilibrium point $E_4 = (\bar{\bar{x}},\bar{\bar{y}},\bar{\bar{z}},0)$ exists and unique in the Int. R_+^3 of xyz-space if and only if there is a positive solution to the following set of equations

$$1 - x - (1+c_1)y - \frac{c_2 z}{c_3+x} = 0 \quad (3.4)$$

$$c_1 x - c_4 z - (c_6+c_7) = 0 \quad (3.5)$$

$$\frac{c_8 x}{c_3+x} + c_9 y - c_{11} = 0 \quad (3.6)$$

From equation (3.5) we have,

$$z = \frac{1}{c_4}\left(c_1 x - (c_6+c_7)\right) \quad (3.7)$$

Also, from equation (3.6) we have,

$$y = \frac{1}{c_9}\left(c_{11} - \frac{c_8 x}{c_3+x}\right) \quad (3.8)$$

Now, by substituting equations (3.7) and (3.8) in equation (3.4) we get:

$$M_1 x^2 + M_2 x + M_3 = 0 \quad (3.9)$$

Where

$M_1 = -c_4 c_9$

$M_2 = c_9(c_4(1-c_3) - c_1 c_2) + c_4(1+c_1)(c_8-c_{11})$

$M_3 = c_3 c_4\left(c_9 - c_{11}(1+c_1)\right) + c_2 c_9(c_6+c_7)$

Note that equation (3.9) has a unique positive root, namely $\bar{\bar{x}}$ provided that:

$$c_9 > c_{11}(1+c_1) \quad (3.10)$$

Substituting the value of $\bar{\bar{x}}$ in (3.7) and (3.8) yield that $z(\bar{\bar{x}}) = \bar{\bar{z}}$ and $y(\bar{\bar{x}}) = \bar{\bar{y}}$ which are positive if the following condition hold:

$$\bar{\bar{x}} > \frac{c_6+c_7}{c_1} \quad (3.11)$$

$$c_{11} > \frac{c_8\bar{\bar{x}}}{c_3+\bar{\bar{x}}} \quad (3.12)$$

Consequently, the infected predator free equilibrium point $E_4 = (\bar{\bar{x}},\bar{\bar{y}},\bar{\bar{z}},0)$ of system (2.2) exists uniquely in the $Int.R_+^3$ of xyz $-$space.

6. The infected-prey-free equilibrium point $E_5 = (\tilde{x}, 0, \tilde{z}, \tilde{w})$ exists and unique in the Int. R_+^3 of xyz-space if and only if there is a positive solution to the following set of equations:

$$1 - x - \frac{c_2 z}{c_3+x} = 0 \quad (3.13)$$

$$\frac{c_8 x}{c_3+x} - c_{10} w - c_{11} = 0 \quad (3.14)$$

$$c_{10} z - (c_{11} + c_{13} + c_{14}) = 0 \quad (3.15)$$

From equation (3.14) we have,

$$\tilde{z} = \frac{c_{11}+c_{13}+c_{14}}{c_{10}} \tag{3.16}$$

Now, by substituting equation (3.16) in equation (3.13) we get:

$$\gamma_1 x^2 + \gamma_2 x + \gamma_3 = 0, \tag{3.17}$$

Where

$$\gamma_1 = -1$$
$$\gamma_2 = 1 - c_3$$
$$\gamma_3 = c_3 - \frac{c_2(c_{11}+c_{13}+c_{14})}{c_{10}}$$

Note that equation (3.17) has a unique positive root, namely $\tilde{x}$ provided that:

$$c_3 > \frac{c_2(c_{11}+c_{13}+c_{14})}{c_{10}} \tag{3.18}$$

Substituting the value of $\tilde{x}$ in (3.14) yield that $w(\tilde{x}) = \widetilde{w} = \frac{\tilde{x}(c_8-c_{11})-c_3c_{11}}{c_{10}(c_3+\tilde{x})}$

which is positive if in addition of condition (3.2) the following conditions hold:

$$\tilde{x} > \frac{c_3c_{11}}{c_8-c_{11}} \tag{3.19}$$

7. The positive (coexistence) equilibrium point $E_6 = (x^*, y^*, z^*, w^*)$ exists if and only if there is a positive solution to the following set of equations

$$1 - x - (1+c_1)y - \frac{c_2 z}{c_3+x} = 0 \tag{3.20}$$

$$c_1 x - c_4 z - c_5 w - (c_6 + c_7) = 0 \tag{3.21}$$

$$\frac{c_8 x}{c_3+x} + c_9 y - c_{10} w - c_{11} = 0 \tag{3.22}$$

$$c_{10} z + c_{12} y - (c_{11} + c_{13} + c_{14}) = 0 \tag{3.23}$$

From equation (3.23) we have,

$$z = \frac{1}{c_{10}}\left((c_{11}+c_{13}+c_{14}) - c_{12}y\right) \tag{3.24}$$

Also, from equation (3.22) we have,

$$w = \frac{1}{c_{10}}\left(\frac{c_8 x}{c_3+x} + c_9 y - c_{11}\right) \tag{3.25}$$

Then by substituting equation (3.24) and (3.25) in (3.20) and (3.21) yield the following two isoclines:

$$g_1(x,y) = 1 - x - (1+c_1)y - \frac{\frac{c_2}{c_{10}}((c_{11}+c_{13}+c_{14})-c_{12}y)}{c_3+x} = 0 \tag{3.26}$$

$$g_2(x,y) = c_1 x - \frac{c_4}{c_{10}}\left((c_{11}+c_{13}+c_{14}) - c_{12}y\right) - \frac{c_5}{c_{10}}\left(\frac{c_8 x}{c_3+x} + c_9 y - c_{11}\right) - (c_6+c_7) = 0 \quad \text{….(3.27)}$$

Now from equation (3.26) we notice that, when $y \to 0$, then $x \to x_1$, where x_1 represents a positive root of the following second order polynomial equation:

$$N_1 x^2 + N_2 x + N_3 = 0 \ , \tag{3.28}$$

Where

$$N_1 = c_{10}$$
$$N_2 = c_3 - 1$$
$$N_3 = c_2(c_{11}+c_{13}+c_{14}) - c_3 c_{10}$$

Straightforward computation shows that equation (3.28) has a unique positive root namely x_1 if the condition (3.18) is hold. Further, from equation (3.27) we notice that, when $y \to 0$, then $x \to x_2$, where x_2 represents a positive root of the following second order polynomial equation:

$$\beta_1 x^2 + \beta_2 x + \beta_3 = 0, \tag{3.29}$$

where

$$\beta_1 = c_1 c_{10}$$
$$\beta_2 = -c_4(c_{11}+c_{13}+c_{14}) + c_1 c_3 c_{10} - c_5(c_8 - c_{11}) - c_{10}(c_6+c_7)$$

$$\beta_3 = -c_3[c_4(c_{13} + c_{14}) + c_{10}(c_6 + c_7) + c_{11}(c_4 - c_5)]$$

Straightforward computation shows that equation (3.29) has a unique positive root namely x_2 if the condition is hold.

$$c_4 > c_5 \tag{3.30}$$

Now, from equation (3.26) we have:

$\frac{dx}{dy} = -\left(\frac{\partial g_1}{\partial y}\right) / \left(\frac{\partial g_1}{\partial x}\right)$. So, $\frac{dx}{dy} > 0$ if one set of the following sets of conditions hold:

$$\left(\frac{\partial g_1}{\partial y}\right) > 0, \left(\frac{\partial g_1}{\partial x}\right) < 0 \;\; OR \;\; \left(\frac{\partial g_1}{\partial y}\right) < 0, \left(\frac{\partial g_1}{\partial x}\right) > 0 \tag{3.31}$$

Further, from (3.27) we notice that

$\frac{dx}{dy} = -\left(\frac{\partial g_2}{\partial y}\right) / \left(\frac{\partial g_2}{\partial x}\right)$. So, $\frac{dx}{dy} < 0$ if one set of the following sets of conditions hold:

$$\left(\frac{\partial g_2}{\partial y}\right) > 0, \left(\frac{\partial g_2}{\partial x}\right) > 0 \;\; OR \;\; \left(\frac{\partial g_2}{\partial y}\right) < 0, \left(\frac{\partial g_2}{\partial x}\right) < 0 \tag{3.32}$$

Then the two isoclines (3.26) and (3.27) intersect at a unique positive point (x^*, y^*), if in addition the condition $x_2 > x_1$ (3.33)

Now, by substituting the value of x^* and y^* in (3.24) and (3.25) yield that $z(y^*) = z^*$ and $w(x^*, y^*) = w^*$ which are positive if and only if the following conditions hold:

$$\frac{c_{11}}{c_9} < y^* < \frac{c_{11}+c_{13}+c_{14}}{c_{12}} \tag{3.34}$$

Accordingly, the positive equilibrium point E_6 exists unique in Int R_+^4, if addition to condition (3.31 -3.34) the isocline $g_1(x, y) = 0$ intersect the x-axis at the positive value namely x_1^*

Local Stability Analysis

In this section, we analyzed the local stability of the model (2.2) around each equilibrium point and discussed through computing the Jacobian matrix $J(x, y, z, w)$ and determined the eigenvalues of system (2.2) at each of them the Jacobian matrix $J(x, y, z, w)$ of the system (2.2) at each of them can be written:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} & \frac{\partial f_3}{\partial w} \\ \frac{\partial f_4}{\partial x} & \frac{\partial f_4}{\partial y} & \frac{\partial f_4}{\partial z} & \frac{\partial f_4}{\partial w} \end{bmatrix} . \tag{4.1}$$

Where f_i ; 1,2,3,4 are given in system (2.2) and

$$\frac{\partial f_1}{\partial x} = 1 - 2x - (1 + c_1)y - \frac{c_3 c_2 z}{(c_3 + x)^2}, \frac{\partial f_1}{\partial y} = -(1 + c_1)x, \frac{\partial f_1}{\partial z} = -\frac{c_2 x}{c_3 + x}, \frac{\partial f_1}{\partial w} = 0,$$

$$\frac{\partial f_2}{\partial x} = c_1 y, \frac{\partial f_2}{\partial y} = c_1 x - c_4 z - c_5 w - (c_6 + c_7), \frac{\partial f_2}{\partial z} = -c_4 y, \frac{\partial f_2}{\partial w} = -c_5 y,$$

$$\frac{\partial f_3}{\partial x} = \frac{c_8 c_3 z}{(c_3 + x)^2}, \frac{\partial f_3}{\partial y} = c_9 z, \quad \frac{\partial f_3}{\partial z} = \frac{c_8 x}{c_3 + x} + c_9 y - c_{10} w - c_{11}, \frac{\partial f_3}{\partial w} = -c_{10} z,$$

$$\frac{\partial f_4}{\partial x} = 0, \frac{\partial f_4}{\partial y} = c_{12} w, \frac{\partial f_4}{\partial z} = c_{10} w, \frac{\partial f_4}{\partial w} = c_{10} z + c_{12} y - (c_{11} + c_{13} + c_{14}).$$

Stability of equilibrium point $E_0 = (0, 0, 0, 0)$

The Jacobian matrix of system (2.2) at E_0 can be written as,

$$J_0 = J(E_0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -(c_6 + c_7) & 0 & 0 \\ 0 & 0 & -c_{11} & 0 \\ 0 & 0 & 0 & -(c_{11} + c_{13} + c_{14}) \end{bmatrix} \quad (4.1)$$

Then the characteristic equation of $J(E_0)$ is given by:

$(1 - \lambda)(c_6 + c_7 + \lambda)(c_{11} + \lambda)((c_{11} + c_{13} + c_{14}) + \lambda) = 0,$
So, the eigenvalues of J_0 are $\lambda_{01} = 1$, $\lambda_{02} = -(c_6 + c_7)$, $\lambda_{03} = -c_{11}$ and $\lambda_{04} = (c_{11} + c_{13} + c_{14})$

Thus, the equilibrium point E_0 is unstable.

Stability of equilibrium point $E_1 = (1, 0, 0, 0)$

The Jacobian matrix of system (2.2) at E_1 can be written as,

$$J_1 = J(E_1) = \begin{bmatrix} -1 & -(1 + c_1) & -\frac{c_2}{c_3 + 1} & 0 \\ 0 & c_1 - (c_6 + c_7) & 0 & 0 \\ 0 & 0 & \frac{c_8}{c_3 + 1} - c_{11} & 0 \\ 0 & 0 & 0 & -(c_{11} + c_{13} + c_{14}) \end{bmatrix} \quad (4.2)$$

Then the characteristic equation of $J(E_1)$ is given by:

$$(1 + \lambda)(c_6 + c_7 - c_1 + \lambda)\left(c_{11} - \frac{c_8}{c_3 + 1} + \lambda\right)((c_{11} + c_{13} + c_{14}) + \lambda) = 0$$

So, the eigenvalues of J_1 are $\lambda_{11} = -1$, $\lambda_{12} = c_1 - (c_6 + c_7)$, $\lambda_{13} = \frac{c_8}{c_3+1} - c_{11}$ and

$\lambda_{14} = (c_{11} + c_{13} + c_{14})$.

Thus, the equilibrium point E_1 is locally asymptotically stable in the. R_+^4, provided that:

$$c_6 + c_7 > c_1 \quad (4.3)$$

$$c_{11} > \frac{c_8}{c_3+1} \quad (4.4)$$

However, it is a saddle point otherwise.

Stability of equilibrium point $E_2 = (\bar{x}, \bar{y}, 0, 0)$

The Jacobian matrix of system (2.2) at E_2 can be written as,

$$J_2 = J(E_2) = [k_{ij}]_{4\times 4} \quad (4.5)$$

Where

$k_{11} = -\frac{c_6 + c_7}{c_1}$, $k_{12} = -\frac{(1 + c_1)(c_6 + c_7)}{c_1}$, $k_{13} = -\frac{c_2(c_6 + c_7)}{c_1 c_3 + c_6 + c_7}$, $k_{14} = 0$,
$k_{21} = \frac{c_1 - (c_6 + c_7)}{1 + c_1}$, $k_{22} = 0$, $k_{23} = -\frac{c_4}{c_1}\left(\frac{c_1 - (c_6 + c_7)}{1 + c_1}\right)$, $k_{24} = -\frac{c_5}{c_1}\left(\frac{c_1 - (c_6 + c_7)}{1 + c_1}\right)$,
$k_{31} = 0$, $k_{32} = 0$, $k_{33} = \frac{c_8(c_6 + c_7)}{c_1 c_3 + c_6 + c_7} + \frac{c_9}{c_1}\left(\frac{c_1 - (c_6 + c_7)}{1 + c_1}\right) - c_{11}$, $k_{34} = 0$, $k_{41} = 0$, $k_{42} = 0$,
$k_{43} = 0$, $k_{44} = \frac{c_{12}}{c_1}\left(\frac{c_1 - (c_6 + c_7)}{1 + c_1}\right) - (c_{11} + c_{13} + c_{14})$.

Then the characteristic equation of $J(E_2)$ is given by:

$[\lambda^2 + B_1\lambda + B_2](k_{33} - \lambda)(k_{44} - \lambda) = 0,$
where:
$B_1 = -k_{11} > 0$
$B_2 = -k_{12}k_{21} > 0$

So, either

$$(k_{33} - \lambda)(k_{44} - \lambda) = 0, \quad (4.6)$$

which gives two of the eigenvalues of $J(E_2)$ by:

$\lambda_{23} = k_{33} < 0$, provided that

$$c_{11} > \frac{c_8(c_6+c_7)}{c_1c_3+c_6+c_7} + \frac{c_9}{c_1}\left(\frac{c_1-(c_6+c_7)}{1+c_1}\right), \tag{4.7}$$

and $\lambda_{24} = k_{44} < 0$, provided that

$$\frac{c_{12}}{c_1}\left(\frac{c_1-(c_6+c_7)}{1+c_1}\right) < (c_{11} + c_{13} + c_{14}) \tag{4.8}$$

Or

$\lambda^2 + B_1\,\lambda + B_2 = 0$

which gives that other two eigenvalues of J_2 with negative real parts which are

$$\lambda_{21} = \frac{1}{2}\left(-B_1 + \sqrt{B_1^2 - 4B_2}\right)$$
$$\lambda_{22} = \frac{1}{2}\left(-B_1 - \sqrt{B_1^2 - 4B_2}\right)$$

So, equilibrium point E_2 is locally asymptotically stable in the . R_+^4 . However, it is unstable otherwise.

Stability of equilibrium point $\boldsymbol{E_3 = (\dot{x}, 0, \dot{z}, 0)}$

The Jacobian matrix of system (2.2) at E_3 can be written as,

$$J_3 = J(E_3) = [z_{ij}]_{4\times 4}, \tag{4.9}$$

Where

$$z_{11} = \dot{x}\left(-1 + \frac{c_2\dot{z}}{(c_3+\dot{x})^2}\right), z_{12} = -(1+c_1)\dot{x},\ z_{13} = -\frac{c_2\dot{x}}{c_3+\dot{x}}, z_{14} = 0, z_{21} = 0,$$

$$z_{22} = c_1\dot{x} - c_4\dot{z} - (c_6+c_7), z_{23} = 0, z_{24} = 0, z_{31} = \frac{c_8c_3\dot{z}}{(c_3+\dot{x})^2}, z_{32} = c_9\dot{z}, z_{33} = 0,$$

$$z_{34} = -c_{10}\dot{z},\ z_{41} = 0,\ z_{42} = 0, z_{43} = 0, z_{44} = c_{10}\dot{z} - (c_{11} + c_{13} + c_{14}).$$

Then the characteristic equation of $J(E_3)$ is given by:

$[\lambda^2 + V_1\,\lambda + V_2]\,(k_{22} - \lambda)(k_{44} - \lambda) = 0,$

where:

$V_1 = -z_{11} > 0$
$V_2 = -z_{13}z_{31} > 0$

So, either

$$(z_{22} - \lambda)(z_{44} - \lambda) = 0 \quad , \tag{4.10}$$

which gives two of the eigenvalues of $J(E_3)$ by:

$\lambda_{32} = z_{22} < 0$, provided that

$$c_1\dot{x} < c_4\dot{z} + (c_6 + c_7), \tag{4.11}$$

And $\lambda_{34} = z_{44} < 0$, provided that

$$c_{10}\dot{z} < (c_{11} + c_{13} + c_{14}) \tag{4.12}$$

Or

$\lambda^2 + V_1\,\lambda + V_2 = 0$

which gives that other two eigenvalues of J_3 with negative real parts which are,

$$\lambda_{31} = \frac{1}{2}\left(-V_1 + \sqrt{V_1^2 - 4V_2}\right)$$
$$\lambda_{33} = \frac{1}{2}\left(-V_1 - \sqrt{V_1^2 - 4V_2}\right)$$

The equilibrium point E_3 is locally asymptotically stable in the . R_+^4 . However, it is unstable otherwise.

Stability of equilibrium point $\boldsymbol{E_4 = (\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{z}}, 0)}$

The Jacobian matrix of system (2.2) at E_4 can be written as,

$J_4 = J(E_4) = [d_{ij}]_{4\times 4}$, (4.13)

Where

$d_{11} = -\bar{\bar{x}} + \dfrac{c_2\bar{\bar{x}}\bar{\bar{z}}}{(c_3 + \bar{\bar{x}})^2}$, $d_{12} = -(1 + c_1)\bar{\bar{x}}, d_{13} = -\dfrac{c_2\bar{\bar{x}}}{c_3 + \bar{\bar{x}}}, d_{14} = 0,\ d_{21} = c_1\bar{\bar{y}}, d_{22} = 0,$

$d_{23} = -c_4\bar{\bar{y}}$, $d_{24} = -c_5\bar{\bar{y}}$, $d_{31} = \dfrac{c_3c_8\bar{\bar{z}}}{(c_3 + \bar{\bar{x}})^2}$, $d_{32} = c_9\bar{\bar{z}}$, $d_{33} = 0$, $d_{34} = -c_{10}\bar{\bar{z}}$, $d_{41} = 0,$

$d_{42} = 0, d_{43} = 0,\ d_{44} = c_{10}\bar{\bar{z}} + c_{12}\bar{\bar{y}} - (c_{11} + c_{13} + c_{14})$

Then the characteristic equation of $J(E_4)$ is given by:

$[\lambda^3 + U_1\lambda^2 + U_2\lambda + U_3]\,(d_{44} - \lambda) = 0,$ (4.14)

Where

$U_1 = -d_{11}$
$U_2 = -(d_{12}d_{21} + d_{13}d_{31} + d_{23}d_{32})$
$U_3 = -(d_{13}d_{21}d_{32} + d_{23}d_{12}d_{31}) + d_{23}d_{11}d_{32}$

So, either

$(d_{44} - \lambda) = 0$, which gives (4.15)

$\lambda_{44} = d_{44} < 0$, provided that

$c_{10}\bar{\bar{z}} + c_{12}\bar{\bar{y}} < (c_{11} + c_{13} + c_{14})$ (4.16)

Or
$[\lambda^3 + U_1\lambda^2 + U_2\lambda + U_3]$, (4.17)

Using Routh Hurwitz criterion equation (4.17) has roots (eigenvalues)with negative real parts if and only if $U_1 > 0, U_3 > 0$ and $U_1U_2 - U_3 > 0$.

Now $U_1 > 0$, provided that

$1 > \frac{c_2\bar{\bar{z}}}{(c_3+\bar{\bar{x}})^2}$ (4.18)

Also, due to condition (4.18) we obtain that $U_3 > 0$ provided that:

$c_1c_2c_9 > c_4(1 + c_1)\frac{c_3c_8}{c_3+\bar{\bar{x}}}$ (4.19)

Further, it is easy to check that:

$U_1U_2 - U_3 = d_{11}d_{12}d_{21} + d_{23}d_{12}d_{31} + d_{13}(d_{21}d_{32} + d_{11}d_{31})$

Clearly, the second terms is positive while the third term is positive under the condition

$d_{21}d_{32} < -d_{11}d_{31}$ (4.20)

Hence $U_1U_2 - U_3 > 0$.
So, all the eigenvalues of $J(E_4)$ have negative real part under the given conditions and hence E_4 is locally asymptotically stable. However, it is unstable otherwise.

Stability of equilibrium point $\boldsymbol{E_5 = (\tilde{x}, 0, \tilde{z}, \tilde{w})}$

The Jacobian matrix of system (2.2) at E_5 can be written as,

$J(E_5) = [r_{ij}]_{4\times 4}$, (4.21)
where

$r_{11} = -\tilde{x} + \dfrac{c_2\tilde{x}\tilde{z}}{(c_3 + \tilde{x})^2}$, $r_{12} = -(1 + c_1)\tilde{x}, r_{13} = -\dfrac{c_2\tilde{x}}{c_3 + \tilde{x}},\ r_{14} = 0, r_{21} = 0,$

$r_{22} = c_1\tilde{x} - c_4\tilde{z} - c_5\tilde{w} - (c_6 + c_7), r_{23} = 0$, $r_{24} = 0$, $r_{31} = \frac{c_3c_8\tilde{z}}{(c_3+\tilde{x})^2}$, $r_{32} = c_9\tilde{z}$, $r_{33} = 0$,

$r_{34} = -c_{10}\tilde{z}$, $r_{41} = 0, r_{42} = c_{12}\tilde{w}, r_{43} = c_{10}\tilde{w}, r_{44} = c_{10}\tilde{z} + c_{12}\tilde{y} - (c_{11} + c_{13} + c_{14})$.

Then the characteristic equation of $J(E_5)$ is given by:

$[\lambda^3 + Q_1\lambda^2 + Q_2\lambda + Q_3]\,(r_{22} - \lambda) = 0,$ (4.22)

where
$Q_1 = -r_{11}$
$Q_2 = -(r_{13}r_{31} + r_{34}r_{43})$
$Q_3 = r_{11}r_{34}r_{43}$
So, either

$(r_{22} - \lambda) = 0$ (4.23)
$\lambda_{22} = r_{22} < 0$
provided that :

$c_1\tilde{x} < c_4\tilde{z} + c_5\tilde{w} + (c_6 + c_7)$ (4.24)

Or

$[\lambda^3 + Q_1 \lambda^2 + Q_2\lambda + Q_3]$, (4.25)

Using Routh Hurwitz criterion equation (4.25) has roots (eigenvalues) with negative real parts if and only if $Q_1 > 0$, $Q_3 > 0$ and $Q_1Q_2 - Q_3 > 0$.

Now, $Q_1 > 0$, $Q_3 > 0$ and $Q_1Q_2 - Q_3 = r_{13}r_{31}r_{11} > 0$, provided that

$1 > \frac{c_2\tilde{z}}{(c_3+\tilde{x})^2}$ (4.26)

So, all the eigenvalues of $J(E_5)$ have negative real part under the given conditions and hence E_5 is locally asymptotically stable. However, it is unstable otherwise.

Stability of Equilibrium point $E_6 = (x^*, y^*, z^*, w^*)$

The Jacobian matrix of system (2.2) at E_5 can be written as,

$J(E_6) = [l_{ij}]_{4\times4}$, (4.27)

where

$l_{11} = -x^* + \frac{c_2x^*z^*}{(c_3 + x^*)^2}$, $l_{12} = -(1 + c_1)x^*$, $l_{13} = -\frac{c_2x^*}{c_3 + x^*}$, $l_{14} = 0$, $l_{21} = c_1y^*$, $l_{22} = 0$,

$l_{23} = -c_4y^*$, $l_{24} = -c_5y^*$, $l_{31} = \frac{c_3c_8z^*}{(c_3 + x^*)^2}$, $l_{32} = c_9z^*$, $l_{33} = 0$, $l_{34} = -c_{10}z^*$, $l_{41} = 0$,

$l_{42} = c_{12}w^*$, $l_{43} = c_{10}w^*$, $l_{44} = 0$.

Then the characteristic equation of $J(E_6)$ is given by:

$\lambda^4 + N_1 \lambda^3 + N_2 \lambda^2 + N_3\lambda + N_4 = 0$, (4.28)

where
$N_1 = -l_{11}$
$N_2 = \rho_1 + \rho_2 + \rho_3 + \rho_4 + \rho_5$
$N_3 = (\rho_6 - l_{11}\rho_1) + \rho_7 + \rho_{10} - l_{11}(\rho_2 + \rho_3)$
$N_4 = (-l_{11}\rho_6 + \rho_1\rho_4) + (\rho_5\rho_3 - \rho_9) - (\rho_8 + l_{11}\rho_{10})$

With

$\rho_1 = -l_{34}l_{43}$, $\rho_2 = -l_{32}l_{23}$, $\rho_3 = -l_{24}l_{42}$, $\rho_4 = -l_{12}l_{21}$, $\rho_5 = -l_{31}l_{13}$,
$\rho_6 = -l_{23}l_{34}l_{42}$, $\rho_7 = -l_{12}l_{23}l_{31} - l_{13}l_{21}l_{32}$, $\rho_8 = l_{12}l_{24}l_{31}l_{43}$, $\rho_9 = l_{13}l_{21}l_{34}l_{42}$,
$\rho_{10} = -l_{24}l_{32}l_{43}$.

Now by using Routh Hurwitz criterion all the eigenvalues, which represent the roots of eq. (4.28), have negative real parts if and only if $N_1 > 0$, $N_3 > 0$, $N_4 > 0$ and $\Delta = (N_1N_2 - N_3)N_3 - N_1^2 N_4 > 0$. Clearly $N_1 > 0$, provided that

$1 > \frac{c_2z^*}{(c_3+x^*)^2}$. (4.29)

Hence $N_3 > 0$, provided that:

$N_1c_{10} > c_4c_{12}y^*$, and (4.30)

$c_1c_2c_9(x^* + c_3) > (1 + c_1)c_3c_4c_8$, (4.31)

while $N_4 > 0$ provided that :

$\frac{c_4c_{12}N_1}{(1+c_1)x^*} < c_1c_{10} < \frac{c_3c_5c_8}{(c_3+x^*)^2}$, and (4.32)

$c_9N_1 > (1 + c_1)c_3c_8x^*$ (4.33)

Straight for word computation shows that:

$$\Delta = l_{11}^2\rho_4(\rho_2 + \rho_3) + l_{11}^2\rho_5(\rho_1 + \rho_2) - \rho_6N_3 + l_{11}^2\rho_9 + l_{11}^2(l_{11}\rho_{10} + \rho_8) + l_{11}\rho_6(l_{11}^2 - (\rho_4 + \rho_5)) + (\rho_7 + \rho_{10})(N_1(\rho_4 + \rho_5) - N_3)$$

Clearly, the first five terms are positive under conditions $(4.29) - (4.31)$ while in addition to conditions $(4.29) - (4.31)$ the last second terms are positive provided that:

$$\frac{N_3}{N_1} < (\rho_4 + \rho_5) < N_1^2 \ . \tag{4.34}$$

Hence $\Delta = (N_1 N_2 - N_3)N_3 - N_1^2 N_4 > 0$.

So, all the eigenvalues of $J(E_6)$ have negative real part under the given conditions and hence E_6 is locally asymptotically stable. However, it is unstable otherwise.

Global stability analysis

In this section the global stability analysis for the equilibrium points, which are locally asymptotically stable of system (2.2) is studied analytically by use the suitable of Lyapunov method as shown in the following theorems.

Theorem (5.1)

Assume that the disease and predator free equilibrium point $E_1 = (1, 0, 0, 0)$ of system (2.2) is locally asymptotically stable in the R_+^4.Then E_1 is globally asymptotically stable provided that the following condition hold:

$$\beta_1 > \beta_2 \tag{5.1}$$

where $\beta_1 = [(x-1) + \frac{1}{2}y]^2 + wy(c_5 - c_{12}) + c_{11}z$ and $\beta_2 = \frac{c_2 z}{c_3 + x} + \frac{1}{4}y^2$.

Proof: Consider the following function

$G_1(x, y, z, w) = (x - 1 - \ln x) + y + z + w$

It is easy to see that $G_1(x, y, z, w) \epsilon\, C^1(R_+^4, R)$, and $G_1(E_1) = 0$, and $G_1(x, y, z, w) > 0$; $\forall (x, y, z, w) \neq E_1$. Now by differentiating G_1 with respect to time t and going some algebraic handling, given that:

$$\frac{dG_1}{dt} = -[(x-1) + \frac{1}{2}y]^2 + y\big(c_1 - (c_6 + c_7)\big) + zy(c_9 - c_4) + wy(c_{12} - c_5) - (c_{11}z + (c_{11} + c_{13} + c_{14})w) + \frac{c_2 z}{c_3 + x} + \frac{1}{4}y^2$$

Now, due to the facts $c_4 > c_9$, $c_5 > c_{12}$ that are mentioned in theorem (2.1) and condition (4.3) we obtain that:

$$\frac{dG_1}{dt} < -[(x-1) + \frac{1}{2}y]^2 - wy(c_5 - c_{12}) - c_{11}z + \frac{c_2 z}{c_3 + x} + \frac{1}{4}y^2 = -\beta_1 + \beta_2$$

Thus, $\frac{dG_1}{dt}$ is negative definite and hence G_1 is Lyapunov function under the condition (5.1). So E_1 is a globally asymptotically stable and then the proof is complete

Theorem (5.2)

Assume that the predator free equilibrium point $E_2 = (\bar{x}, \bar{y}, 0, 0)$ of system (2.2) is locally asymptotically stable in the R_+^4.Then E_2 is globally asymptotically stable provided that the following conditions hold:

$$\bar{\beta}_1 > \bar{\beta}_2 \tag{5.2}$$

where $\bar{\beta}_1 = [(x - \bar{x}) + \frac{1}{2}(y - \bar{y})]^2 + (c_8 - c_2)\frac{xz}{c_3 + x} + wy(c_{12} - c_5) + zy(c_9 - c_4) - c_{11}z + w(-c_{12}\bar{y} + (c_{11} + c_{13} + c_{14})$ and $\bar{\beta}_2 = \frac{c_2 \bar{x} z}{c_3 + x} + c_4 \bar{y} z + c_5 w \bar{y} + \frac{1}{4}(y - \bar{y})^2$.

Proof: Consider the following function

$$G_2(x, y, z, w) = \left(x - \bar{x} - \bar{x}\ln\frac{x}{\bar{x}}\right) + (y - \bar{y} - \bar{y}\ln\frac{y}{\bar{y}}) + z + w$$

It is easy to see that $G_2(x, y, z, w) \epsilon\, C^1(R_+^4, R)$, and $G_2(E_2) = 0$, and $G_2(x, y, z, w) > 0$; $\forall (x, y, z, w) \neq E_2$. Now by differentiating G_2 with respect to time t and going some algebraic handling, given that:

$$\frac{dG_2}{dt} = -(x - \bar{x})^2 - (x - \bar{x})(y - \bar{y}) - (x - \bar{x})\frac{c_2 \bar{x} z}{c_3 + x} - c_5 w(y - \bar{y}) - c_4 z(y - \bar{y}) + \frac{c_8 x z}{c_3 + x} + c_9 yz - c_{11}z + c_{12}wy - (c_{11} + c_{13} + c_{14})w \pm \frac{1}{4}(y - \bar{y})^2$$

$$\frac{dG_2}{dt} < -[(x - \bar{x}) + \frac{1}{2}(y - \bar{y})]^2 + (c_8 - c_2)\frac{xz}{c_3 + x} + \frac{c_2 \bar{x} z}{c_3 + x} + wy(c_{12} - c_5) + zy(c_9 - c_4) - c_{11}z + w(c_{12}\bar{y} - (c_{11} + c_{13} + c_{14})) + c_4 \bar{y} z + c_5 w \bar{y} + \frac{1}{4}(y - \bar{y})^2 = -\bar{\beta}_1 + \bar{\beta}_2$$

Thus, $\frac{dG_2}{dt}$ is negative definite and hence G_2 is Lyapunov function under the conditions mentioned in theorem (2.1), $(5.2) and\ (4.8)$. So E_2 is a globally asymptotically stable and then the proof is complete

Theorem (5.3)

Assume that the disease free equilibrium point $E_3 = (\dot{x}, 0, \dot{z}, 0)$ of system (2.2) is locally asymptotically stable in the R_+^4.Then E_3 is globally asymptotically stable provided that the following conditions hold:

$$\dot{\beta}_1 > \dot{\beta}_2 \tag{5.3}$$

$$\left[\frac{\left(c_2\dot{x}+\frac{c_1+1}{c_1}c_8c_3-c_2c_3\right)}{c_3(c_3+\dot{x})}\right]^2 < 4\left(1-\frac{c_2\dot{z}}{c_3(c_3+\dot{x})}\right) , \tag{5.4}$$

$$\frac{c_2\dot{z}}{c_3(c_3+\dot{x})} < 1 \tag{5.5}$$

where $\dot{\beta}_1 = \left[\sqrt{\left(1-\frac{c_2\dot{z}}{c_3(c_3+\dot{x})}\right)}(x-\dot{x}) + (z-\dot{z})\right]^2 - \frac{1+c_1}{c_1}[zy(c_9-c_4) + wy(c_{12}-c_5) - (c_6+c_7)y] - \frac{1+c_1}{c_1} w\left(c_{10}\dot{z} - (c_{11}+c_{13}+c_{14})\right)$ and $\dot{\beta}_2 = (1+c_1)\dot{x}y + (z-\dot{z})^2$

Proof: Consider the following function

$$G_3(x,y,z,w) = \left(x - \dot{x} - \dot{x}\ln\frac{x}{\dot{x}}\right) + \dot{a}_1 y + \dot{a}_2\left(z - \dot{z} - \dot{z}\ln\frac{z}{\dot{z}}\right) + \dot{a}_3 w$$

where $\dot{a}_i$, i=1,2,3 are positive constant to be bent on. It is easy to see that $G_3(x,y,z,w) \epsilon\, C^1(R_+^4, R)$, and $G_3(E_3) = 0$, and $G_3(x,y,z,w) > 0 ; \forall (x,y,z,w) \neq E_3$. Now by differentiating G_3 with respect to time t and going some algebraic handling, given that:

$$\frac{dG_3}{dt} = -\left(1 - \frac{c_2\dot{z}}{(c_3+x)(c_3+\dot{x})}\right)(x-\dot{x})^2 + (x-\dot{x})y(\dot{a}_1c_1 - (1+c_1)) + \dot{a}_1c_1\dot{x}y - \dot{a}_1(c_6+c_7)y + zy(\dot{a}_2c_9 - \dot{a}_1c_4)$$
$$+ wy(\dot{a}_3c_{12} - \dot{a}_1c_5) + \frac{(c_2\dot{x} + \dot{a}_2c_8c_3 - c_2c_3)}{(c_3+x)(c_3+\dot{x})}(x-\dot{x})(z-\dot{z}) - \dot{a}_2c_9y\dot{z} + c_{10}wz(\dot{a}_3 - \dot{a}_2) + \dot{a}_3\, c_{10}w\dot{z}$$
$$- \dot{a}_3(c_{11}+c_{13}+c_{14})w$$

So by choosing the constants

$\dot{a}_1 = \dot{a}_2 = \dot{a}_3 = \frac{1+c_1}{c_1}$, we get:

$$\frac{dG_3}{dt} = -\left(1 - \frac{c_2\dot{z}}{(c_3+x)(c_3+\dot{x})}\right)(x-\dot{x})^2 + (1+c_1)\dot{x}y - \frac{1+c_1}{c_1}(c_6+c_7)y + \frac{\left(c_2\dot{x} + \frac{1+c_1}{c_1}c_8c_3 - c_2c_3\right)}{(c_3+x)(c_3+\dot{x})}(x-\dot{x})(z-\dot{z})$$
$$+ \frac{1+c_1}{c_1}zy(c_9-c_4) + \frac{1+c_1}{c_1}wy(c_{12}-c_5) - \frac{1+c_1}{c_1}c_9y\dot{z} + \frac{1+c_1}{c_1}\left(c_{10}\dot{z} - (c_{11}+c_{13}+c_{14})\right)w \pm (z-\dot{z})^2$$

$$\frac{dG_3}{dt} < -\left[\sqrt{\left(1-\frac{c_2\dot{z}}{c_3(c_3+\dot{x})}\right)}(x-\dot{x}) + (z-\dot{z})\right]^2 + \frac{1+c_1}{c_1}[zy(c_9-c_4) + wy(c_{12}-c_5) - (c_6+c_7)y] + \frac{1+c_1}{c_1}w\left(c_{10}\dot{z} - (c_{11}+c_{13}+c_{14})\right) + (1+c_1)\dot{x}y + (z-\dot{z})^2 = -\dot{\beta}_1 + \dot{\beta}_2$$

Thus, $\frac{dG_3}{dt}$ is negative definite and hence G_3 is Lyapunov function under the conditions mentioned in theorem (2.1), $(5.3)-(5.5) and\ (4.12)$. So E_3 is a globally asymptotically stable and then the proof is complete ■

Theorem (5.4)

Assume that the infected predator free equilibrium point $E_4 = (\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{z}}, 0)$ of system (2.2) is locally asymptotically stable in the R_+^4.Then E_4 is globally asymptotically stable provided that the following conditions hold:

$$y > \bar{\bar{y}} \quad , \bar{\bar{\beta}}_1 > \bar{\bar{\beta}}_2 \quad , \tag{5.6}$$

$$\frac{c_2\bar{\bar{z}}}{c_3(c_3+\bar{\bar{x}})} < 1 \tag{5.7}$$

$$2\left(1-\frac{c_2\bar{\bar{z}}}{c_3(c_3+\bar{\bar{x}})}\right) > 1, \quad 2\left(\frac{c_2\bar{\bar{z}}}{c_3(c_3+\bar{\bar{x}})}\right) > \left(\frac{-c_2c_3+c_2\bar{\bar{x}}+c_8c_3}{(c_3(c_3+\bar{\bar{x}}))}\right)^2 , \tag{5.8}$$

$$(c_9-c_4)^2 < 4 \tag{5.9}$$

where $\bar{\bar{\beta}}_1 = \left[\sqrt{\frac{1}{2}\left(1-\frac{c_2\bar{\bar{z}}}{c_3(c_3+\bar{\bar{x}})}\right)}(x-\bar{\bar{x}}) + (y-\bar{\bar{y}})\right]^2 + \left[\sqrt{\frac{1}{2}\left(1-\frac{c_2\bar{\bar{z}}}{c_3(c_3+\bar{\bar{x}})}\right)}(x-\bar{\bar{x}}) - (z-\bar{\bar{z}})\right]^2 - \left(c_{10}\bar{\bar{z}} + c_{12}\bar{\bar{y}} - (c_{11}+c_{13}+c_{14})\right)w + w(y-\bar{\bar{y}})(c_5-c_{12})$

$\bar{\bar{\beta}}_2 = [(y-\bar{\bar{y}}) + (z-\bar{\bar{z}})]^2$.

Proof: Consider the following function

$$G_4(x,y,z,w) = \left(x-\bar{\bar{x}}-\bar{\bar{x}}\ln\frac{x}{\bar{\bar{x}}}\right) + \bar{\bar{a}}_1\left(y-\bar{\bar{y}}-\bar{\bar{y}}\ln\frac{y}{\bar{\bar{y}}}\right) + \bar{\bar{a}}_2\left(z-\bar{\bar{z}}-\bar{\bar{z}}\ln\frac{z}{\bar{\bar{z}}}\right) + \bar{\bar{a}}_3 w$$

Where $\bar{\bar{a}}_i, i$=1,2,3 are positive constant to be bent on. It is easy to see that $G_4(x,y,z,w)\epsilon\, C^1(R_+^4,R)$, and $G_4(E_4)=0$, and $G_4(x,y,z,w)>0\,;\forall(x,y,z,w)\neq E_4$. Now by differentiating G_4 with respect to time t and going some algebraic handling, given that:

$$\frac{dG_4}{dt} = -\left(1-\frac{c_2\bar{\bar{z}}}{(c_3+x)(c_3+\bar{\bar{x}})}\right)(x-\bar{\bar{x}})^2 + (x-\bar{\bar{x}})(y-\bar{\bar{y}})(\bar{\bar{a}}_1c_1-(1+c_1)) + \frac{(c_2\bar{\bar{x}}+\bar{\bar{a}}_2c_8c_3-c_2c_3)}{(c_3+x)(c_3+\bar{\bar{x}})}(x-\bar{\bar{x}})(z-\bar{\bar{z}}) + (z-\bar{\bar{z}})(y-\bar{\bar{y}})(\bar{\bar{a}}_2c_9-\bar{\bar{a}}_1c_4) + w(y-\bar{\bar{y}})(\bar{\bar{a}}_3c_{12}-\bar{\bar{a}}_1c_5) + c_{10}wz(\bar{\bar{a}}_2-\bar{\bar{a}}_3) + \bar{\bar{a}}_2\,c_{10}w\bar{\bar{z}} + \bar{\bar{a}}_3c_{12}w\bar{\bar{y}} - \bar{\bar{a}}_3(c_{11}+c_{13}+c_{14})w \pm (y-\bar{\bar{y}})^2 \pm (z-\bar{\bar{z}})^2$$

So by choosing the constants

$\bar{\bar{a}}_1 = \bar{\bar{a}}_2 = \bar{\bar{a}}_3 = 1$,

$$\frac{dG_4}{dt} < -\left[\sqrt{\frac{1}{2}\left(1-\frac{c_2\bar{\bar{z}}}{c_3(c_3+\bar{\bar{x}})}\right)}(x-\bar{\bar{x}}) + (y-\bar{\bar{y}})\right]^2 - \left[\sqrt{\frac{1}{2}\left(1-\frac{c_2\bar{\bar{z}}}{c_3(c_3+\bar{\bar{x}})}\right)}(x-\bar{\bar{x}}) - (z-\bar{\bar{z}})\right]^2 + \left(c_{10}\bar{\bar{z}} + c_{12}\bar{\bar{y}} - (c_{11}+c_{13}+c_{14})\right)w + [(y-\bar{\bar{y}})+(z-\bar{\bar{z}})]^2 + w(y-\bar{\bar{y}})(c_{12}-c_5) = -\bar{\bar{\beta}}_1 + \bar{\bar{\beta}}_2$$

Thus, $\frac{dG_4}{dt}$ is negative definite and hence G_4 is Lyapunov function under the conditions mentioned in theorem (2.1), (5.6) – (5.9) and (4.16). So E_4 is a globally asymptotically stable and then the proof is complete ■

Theorem (5.5)

Assume that the infected prey free equilibrium point $E_5 = (\tilde{x},0,\tilde{z},\widetilde{w})$ of system (2.2) is locally asymptotically stable in the R_+^4.Then E_5 is globally asymptotically stable provided that the following conditions hold:

$$w > \widetilde{w},\quad \tilde{\beta}_1 > \tilde{\beta}_2 \ , \tag{5.10}$$

$$\left(c_2\tilde{x} + \frac{c_4}{c_9}\frac{1+c_1}{c_1}c_8c_3 - c_2c_3\right)^2 > 4\left(1-\frac{c_2\tilde{z}}{c_3(c_3+\tilde{x})}\right) , \tag{5.11}$$

$$\frac{c_2\tilde{z}}{c_3(c_3+\tilde{x})} < 1, \tag{5.12}$$

$$c_5c_9 > c_4c_{12}. \tag{5.13}$$

where

$$\tilde{\beta}_1 = \left[\sqrt{\left(1-\frac{c_2\tilde{z}}{c_3(c_3+\tilde{x})}\right)}(x-\tilde{x}) + (z-\tilde{z})\right]^2 - \frac{1+c_1}{c_1}y(w-\widetilde{w})(c_4c_{12}-c_9c_5) - \frac{1+c_1}{c_1}y\left(c_1\tilde{x}-c_4\tilde{z}-c_5\widetilde{w}-(c_6+c_7)\right)$$

$$\tilde{\beta}_2 = (z-\tilde{z})^2$$

Proof: Consider the following function

$$G_5(x,y,z,w) = \left(x-\tilde{x}-\tilde{x}\ln\frac{x}{\tilde{x}}\right) + \tilde{a}_1y + \tilde{a}_2\left(z-\tilde{z}-\tilde{z}\ln\frac{z}{\tilde{z}}\right) + \tilde{a}_3\left(w-\widetilde{w}-\widetilde{w}\ln\frac{w}{\widetilde{w}}\right)$$

Where $\tilde{a}_i, i$=1,2,3 are positive constant to be bent on. It is easy to see that ,$G_5(x,y,z,w)\epsilon\, C^1(R_+^4,R)$, and $G_5(E_5)=0$, and $G_5(x,y,z,w)>0\,;\forall(x,y,z,w)\neq E_5$ .Now, by differentiating G_5 with respect to time t and going some algebraic handling, given that:

$$\frac{dG_5}{dt} = -\left(1-\frac{c_2\tilde{z}}{(c_3+x)(c_3+\tilde{x})}\right)(x-\tilde{x})^2 + (x-\tilde{x})y(\tilde{a}_1c_1-(1+c_1)) + \tilde{a}_1y\left(c_1\tilde{x}-c_4\tilde{z}-c_5\widetilde{w}-(c_6+c_7)\right) + \frac{(c_2\tilde{x}+\tilde{a}_2c_8c_3-c_2c_3)}{(c_3+x)(c_3+\tilde{x})}(x-\tilde{x})(z-\tilde{z}) + (z-\tilde{z})y(\tilde{a}_2c_9-\tilde{a}_1c_4) + y(w-\widetilde{w})(\tilde{a}_3c_{12}-\tilde{a}_1c_5) + c_{10}(w-\widetilde{w})(z-\tilde{z})(\tilde{a}_3-\tilde{a}_2)$$

So by choosing the constants

$\tilde{a}_1 = \frac{1+c_1}{c_1}$, $\tilde{a}_2 = \tilde{a}_3 = \frac{c_4}{c_9}\frac{1+c_1}{c_1}$, we get ;

$$\frac{dG_5}{dt} < -\left(1-\frac{c_2\tilde{z}}{c_3(c_3+\tilde{x})}\right)(x-\tilde{x})^2 + \frac{1+c_1}{c_1}y\left(c_1\tilde{x} - c_4\tilde{z} - c_5\tilde{w} - (c_6+c_7)\right) + \frac{(x-\tilde{x})(z-\tilde{z})}{c_3(c_3+\tilde{x})}\left(c_2\tilde{x} + \frac{c_4}{c_9}\frac{1+c_1}{c_1}c_8c_3 - c_2c_3\right) + \frac{1+c_1}{c_1}y(w-\tilde{w})(c_4c_{12} - c_9c_5) \pm (z-\tilde{z})^2$$

$$\frac{dG_5}{dt} < -\left[\sqrt{\left(1-\frac{c_2\tilde{z}}{c_3(c_3+\tilde{x})}\right)}(x-\tilde{x}) + (z-\tilde{z})\right]^2 + \frac{1+c_1}{c_1}y(w-\tilde{w})(c_4c_{12} - c_9c_5)$$

$$+(z-\tilde{z})^2 + \frac{1+c_1}{c_1}y\left(c_1\tilde{x} - c_4\tilde{z} - c_5\tilde{w} - (c_6+c_7)\right) = -\tilde{\beta}_1 + \tilde{\beta}_2$$

Thus, $\frac{dG_5}{dt}$ is negative definite and hence G_5 is Lyapunov function under the conditions $(5.10)-(5.13) and$ (4.24). So E_5 is a globally asymptotically stable and then the proof is complete

Theorem (5.6)

Assume that the positive equilibrium point $E_6 = (x^*, y^*, z^*, w^*)$ of system (2.2) is locally asymptotically stable. Then E_6 is globally asymptotically stable in the R_+^4 . provided that the following conditions hold:

$$\frac{c_2z^*}{c_3(c_3+x^*)} < 1 \quad , \tag{5.14}$$

$$\left[\frac{\left(c_2x^* + \frac{c_4}{c_9}\frac{1+c_1}{c_1}c_8c_3 - c_2c_3\right)}{c_3(c_3+x^*)}\right]^2 < 4\left(1-\frac{c_2z^*}{c_3(c_3+x^*)}\right), \tag{5.15}$$

$$c_{10}\frac{1+c_1}{c_1}\left(\frac{c_5}{c_{12}} - \frac{c_4}{c_9}\right) < 2 , \tag{5.16}$$

$$\beta_1^* > \beta_2^* . \tag{5.17}$$

Where

$$\beta_1^* = \left[\sqrt{\left(1-\frac{c_2z^*}{c_3(c_3+x^*)}\right)}(x-x^*) - (z-z^*)\right]^2 + (w-w^*)^2$$

$$\beta_2^* = [(z-z^*) + (w-w^*)]^2$$

Proof: Consider the following function

$$G_6(x,y,z,w) = \left(x - x^* - x^*\ln\frac{x}{x^*}\right) + a_1^*\left(y - y^* - y^*\ln\frac{x}{y^*}\right) + a_2^*(z - z^* - z^*\ln\frac{z}{z^*}) + a_3^*(w - w^* - w^*\ln\frac{w}{w^*})$$

where a_i^*, i=1,2,3 are positive constant to be bent on. It is easy to see that $,G_6(x,y,z,w)\epsilon\, C^1(R_+^4, R)$, and $G_6(E_6) = 0$, and $G_6(x,y,z,w) > 0$; $\forall(x,y,z,w) \neq E_6$. Now by differentiating G_6 with respect to time t and going some algebraic handling, given that:

$$\frac{dG_6}{dt} = -\left(1-\frac{c_2z^*}{(c_3+x)(c_3+x^*)}\right)(x-x^*)^2 + c_{10}(w-w^*)(z-z^*)(a_3^*-a_2^*) + (x-x^*)(y-y^*)(a_1^*c_1 - (1+c_1)) + (y-y^*)(z-z^*)(a_2^*c_9 - a_1^*c_4) + \frac{(c_2x^* + a_2^*c_8c_3 - c_2c_3)}{(c_3+x)(c_3+x^*)}(x-x^*)(z-z^*) + (y-y^*)(w-w^*)(a_3^*c_{12} - a_1^*c_5)$$

So by choosing the constants

$a_1^* = \frac{1+c_1}{c_1}$, $a_2^* = \frac{c_4}{c_9}\frac{1+c_1}{c_1}$, $a_3^* = \frac{c_5}{c_{12}}\frac{1+c_1}{c_1}$, we get:

$$\frac{dG_6}{dt} = -\left(1-\frac{c_2z^*}{(c_3+x)(c_3+x^*)}\right)(x-x^*)^2 + c_{10}\frac{1+c_1}{c_1}(w-w^*)(z-z^*)\left(\frac{c_5}{c_{12}} - \frac{c_4}{c_9}\right) + \frac{\left(c_2x^* + \frac{c_4}{c_9}\frac{1+c_1}{c_1}c_8c_3 - c_2c_3\right)}{(c_3+x)(c_3+x^*)}(x-x^*)(z-z^*) \pm (z-z^*)^2 \pm (w-w^*)^2$$

$$\frac{dG_6}{dt} < -\left[\sqrt{\left(1-\frac{c_2 z^*}{c_3(c_3+x^*)}\right)}\,(x-x^*)-(z-z^*)\right]^2 - (w-w^*)^2 + [(z-z^*)+(w-w^*)]^2 = -\beta_1^* + \beta_2^*$$

Thus, $\frac{dG_6}{dt}$ is negative definite and hence G_6 is Lyapunov function under the conditions (5.14) – (5.17). So E_6 is a globally asymptotically stable and then the proof is complete

Numerical simulation

In this section, we confirmed our obtained results in the previous sections numerically by using Runge Kutta method along with predictor corrector method. Note that, we use turbo C++ in programming and matlab in plotting and then discuss our obtained results. The system (2.2) is studied numerically for different sets of parameters and different sets of initial points. The objectives of this study are: first investigate the effect of varying the value of each parameter on the dynamical behavior of system (2.2) and second confirm our obtained analytical results. It is observed that, for the following set of hypothetical parameters:

$$\left.\begin{array}{l} c_1 = 0.5, c_2 = 0.4, c_3 = 0.4, c_4 = 0.5, c_5 = 0.3, c_6 = 0.01, c_7 = 0.1 \\ c_8 = 0.3, c_9 = 0.4, c_{10} = 0.5, c_{11} = 0.01, c_{12} = 0.2, c_{13} = 0.01, c_{14} = 0.1 \end{array}\right\} \quad (6.1)$$

.

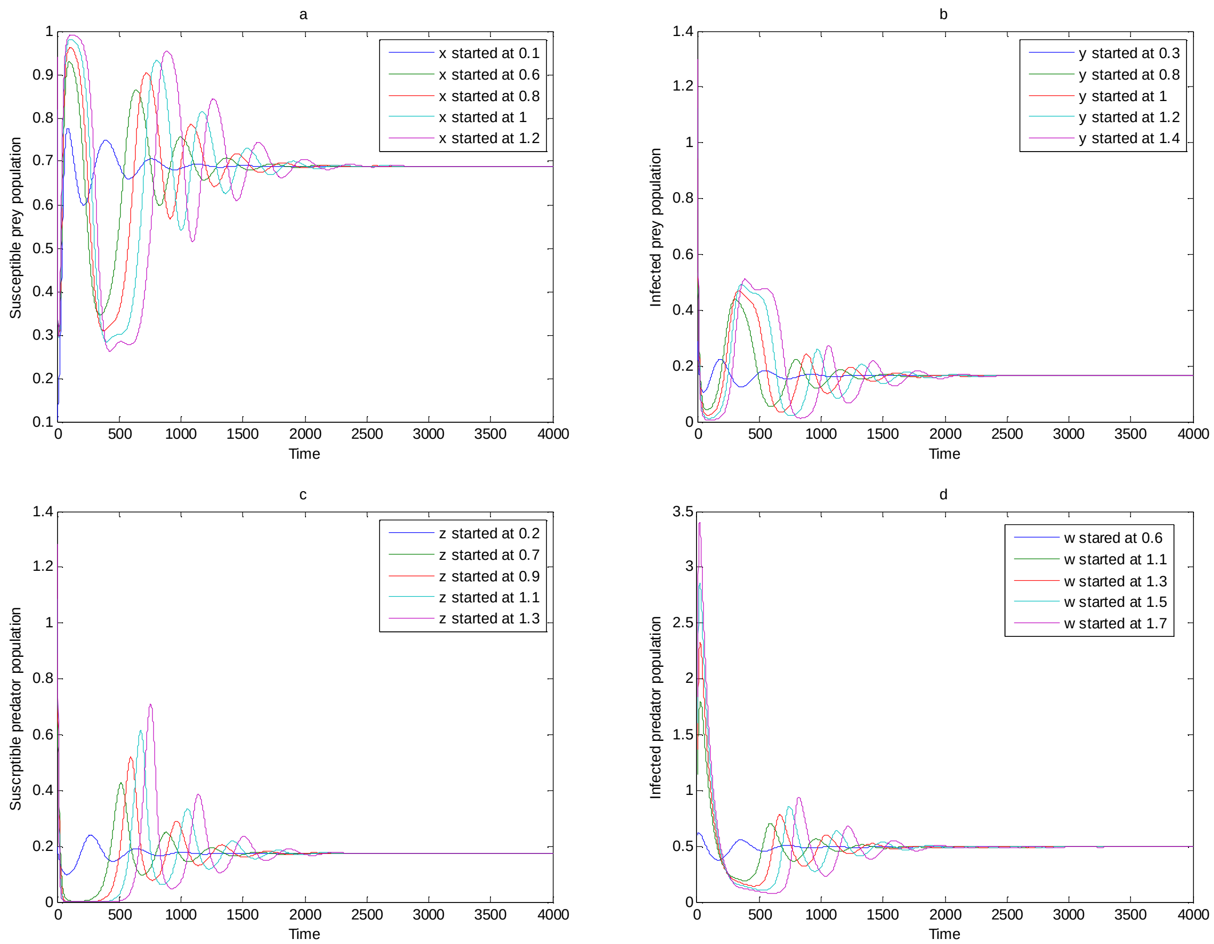

Fig 1 Time series of the solution of system (2.2) that started from five different initial points (0.6 ,0.8 , 0.7 , 1.1) , (0.8 , 1 , 0.9, 1.3), (1,1.2,1.1,1.5), and (1.2,1.4,1.3,1.7) for the data given in (6.1). (a) Time series of the trajectories of susceptible prey x , (b) Time series of the trajectories of infected prey y, (c) Time series of the trajectories of susceptible predator z, (d) Time series of the trajectories of infected predator w.

Clearly, figure (6.1) shows that system (2.2) approaches asymptotically to the positive equilibrium point E_6 = (0.688, 0.164, 0.174, 0.491) starting from five different initial points and this is confirming our obtained analytical results.

Now, in order to discuss the effect of the parameters values of system (2.2) on the dynamical behavior of the system, the system is solved numerically for the data given in (6.1) with varying one parameter at each time and sometime two parameters the obtained results are given below.

The effect of varying the infection rate of prey in the range $0 < c_1 < 0.37$ keeping other parameters as data given in (6.1) , causes extinction in the infected prey and the system will approach to the infected prey free equilibrium point as shown in the following figure. However for $0.37 < c_1 < 1.5$, it is observed that system (2.2) still approach asymptotically to the positive equilibrium point.

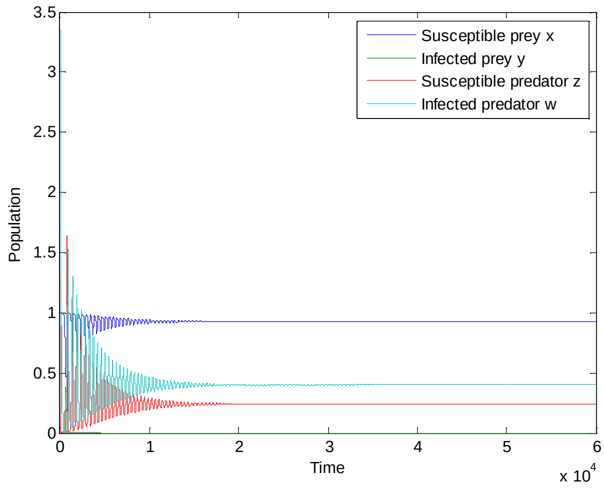

Fig 2 Time series of the solution of system (2.2) approaches asymptotically to the infected prey free equilibrium point $E_5 = (0.92, 0, 0.24, 0.39)$ for the data given in (6.1) with $c_1 = 0.3$.

The effect of varying the predation rate on susceptible prey in the range $0.3 < c_2 < 1.45$ keeping other parameters as data given in (6.1) is studied, it is observed that system (2.2) still approach asymptotically to the positive equilibrium point, while increasing this parameter further $1.45 < c_2$ causes extinction in the infected prey and the system will approach the infected prey free equilibrium point. On other hand varying the half saturation rate in the range $0 < c_3 < 1.5$ keeping other parameters as data given in (6.1) is studied, it is observed that system (2.2) still approach asymptotically to the positive equilibrium point. Moreover, varying the predation rate on infected prey in the range $0.4 < c_4 < 0.97$ keeping other parameters as data given in (6.1) is studied; it is observed that system (2.2) still approach asymptotically to the positive equilibrium point, while increasing this parameter further $0.98 < c_4 < 1.5$ cause's extinction in the infected prey and the system will approach to the infected prey free equilibrium point.

The effect of varying the predation rate on infected prey in the range $0.2 < c_5 < 0.58$ keeping other parameters as data given in (6.1); it is observed that system (2.2) still approach asymptotically to the positive equilibrium point, however for $0.59 < c_5 < 1.5$ cause's extinction in the infected prey and the system will approach the infected prey free equilibrium point.

The effect of varying the death rate of the infected prey due to disease, in the range $0 < c_6 < 0.124$ keeping other parameters as data given in (6.1) is studied, it is observed that system (2.2) still approach asymptotically to the positive equilibrium point, however increasing this parameter further $0.124 < c_6 < 1$ causes extinction in the infected prey and the system will approach the infected prey free equilibrium point.

The effect of varying the harvesting rate of infected prey, in the range $0 < c_7 < 0.214$ keeping other parameters as data given in (6.1) is studied, it is observed that system (2.2) still approach asymptotically to the positive equilibrium point, however increasing this parameter further $0.215 < c_7 < 1$ causes extinction in the infected prey and the system will approach the infected prey free equilibrium point.

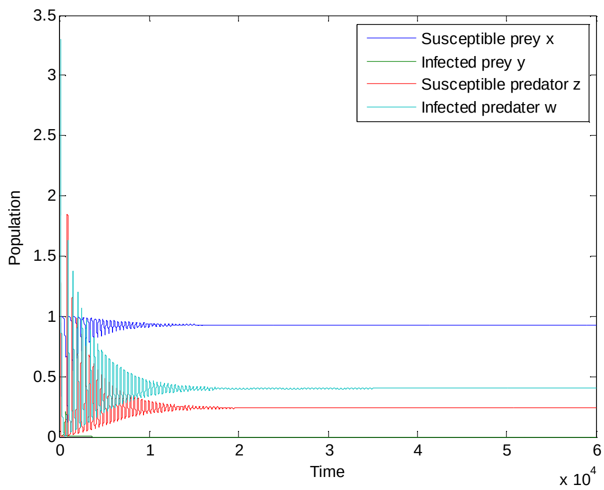

Fig 3 Time series of the solution of system (2.2) approaches asymptotically to the infected prey free equilibrium point $E_5 = (0.92, 0, 0.24, 0.39)$ for the data given in (6.1) with $c_7 = 0.3$.

Now by varying the conversion rate of the susceptible prey from susceptible predator, in the range $0 < c_8 < 0.4$ keeping other parameters as data given in (6.1) is studied; it is observed that system (2.2) still approach asymptotically to the positive equilibrium point, however $c_8 < 0.012$ and $c_1 < 0.1$ keeping other parameters as data given in (6.1) is studied; it is observed that the solution of system (2.2) approaches asymptotically to the axial equilibrium point as shown in the following figure.

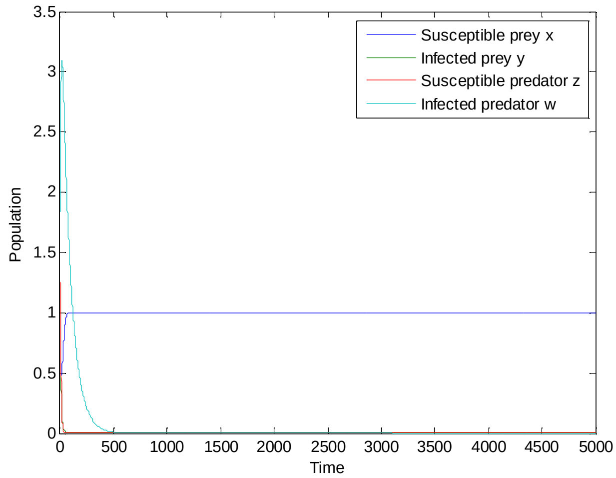

Fig 4 Time series of the solution of system (2.2) approaches asymptotically to the axial equilibrium point $E_1 = (1,0,0,0)$ for the data given in (2.1) with $c_1 = 0.09$ and $c_8 = 0.01$.

The effect of varying the conversion rate of the infected prey from susceptible predator, in the range $0 < c_9 < 0.5$ keeping other parameters as data given in (6.1) is studied; it is observed that system (2.2) still approach asymptotically to the positive equilibrium point.

The effect of varying the infection rate of predator in the range $0.1 < c_{10} < 0.35$, causes extinction in the infected prey and the system will approach the infected prey free equilibrium point as shown in the following figure. However $0.35 < c_{10} < 0.95$ keeping other parameters as data given in (6.1) is studied, it is observed that system (2.2) still approach asymptotically to the positive equilibrium point.

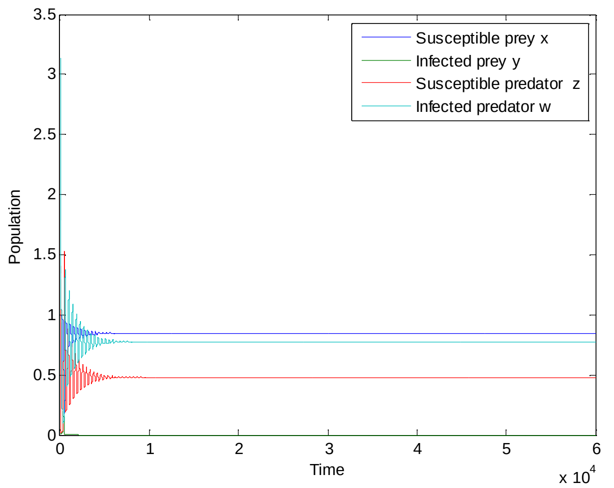

Fig 5 Time series of the solution of system (2.2) approaches asymptotically to the infected prey free equilibrium point $E_5 = (0.84, 0, 0.48, 0.77)$ for the data given in (6.1) with $c_{10} = 0.25$.

Similarly, for the data given by Eq(6.1), the effect of varying the death rate of the predator, in the range $0 < c_{11} < 0.168$ keeping other parameters as data given in (6.1) is studied; it is observed that system (2.2) still approach asymptotically to the positive equilibrium point, while for $0.168 \leq c_{11} < 0.2$ causes extinction in the infected prey and the system will approach the infected prey free equilibrium point, further for $c_{11} = 0.2$ the solution of the system (2.2) approaches to the disease free equilibrium point as show in the Fig(6.6a) ;additional for $0.2 < c_{11} \leq 0.31$ causes extinction in the infected predator and the system will approach the infected predator free equilibrium point as show in the Fig(6.6b), finally $0.31 < c_{11} \leq 1$ the solution of the system (2.2) approaches to the predator free equilibrium point as show in the Fig(6.6c).

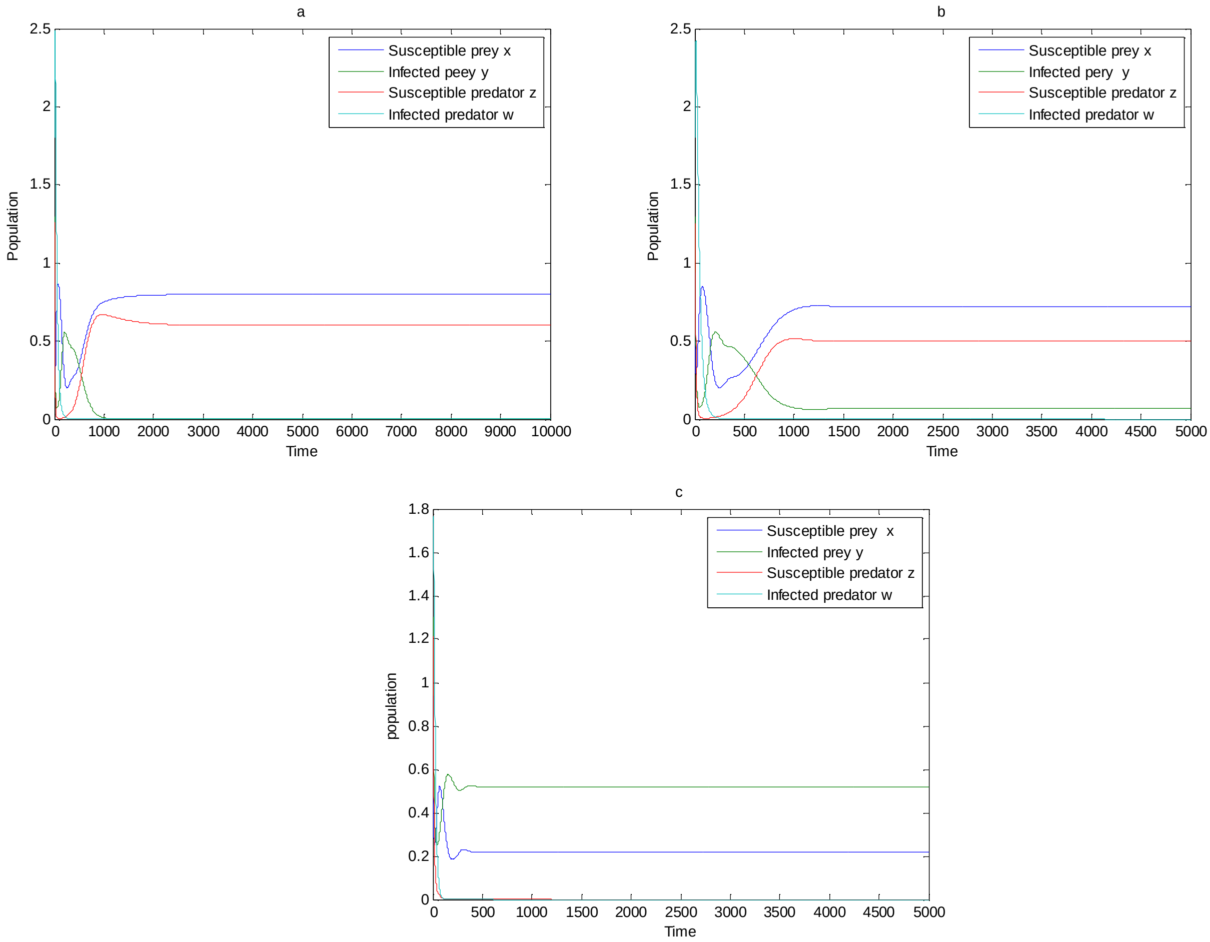

Fig 6 Time series of the solution of system (2.2) for the data given in (6.1) with different value of c_{11} : (a) globally asymptotically stable of the disease free equilibrium point $E_3 = (0.8, 0, 0.6, 0)$ for $c_{11} = 0.2$, (b) globally asymptotically stable of the infected predator free equilibrium point $E_4 = (0.719, 0.067, 0.499, 0)$ for $c_{11} = 0.25$, (c) globally asymptotically stable predator free equilibrium point $E_2 = (0.22, 0.52, 0, 0)$ for $c_{11} = 0.6$.

The effect of varying the conversion rate of the infected prey from predator, in the range $0 < c_{12} \le 0.3$ keeping other parameters as data given in (6.1) is studied; it is observed that system (2.2) still approach asymptotically to the positive equilibrium point. On the other hand varying the effect of the death rate of the infected predator due to disease, in the range $0 < c_{13} < 0.09$ keeping other parameters as data given in (6.1) is studied; it is observed that system (2.2) still approach asymptotically to the positive equilibrium point, however for $0.1 \le c_{13} \le 0.41$ causes extinction in the infected prey and the system will approach the infected prey free equilibrium point as shown in the following figure.

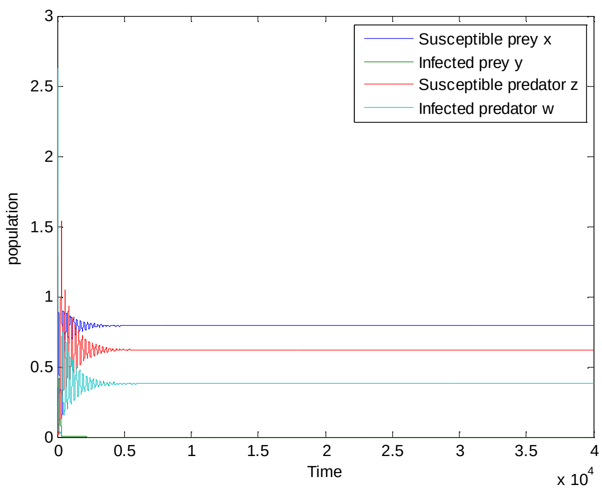

Fig 7 Time series of the solution of system (2.2) approaches asymptotically to the infected prey free equilibrium point $E_5 = (0.79, 0, 0.62, 0.37)$ for the data given in (6.1) with $c_{13} = 0.2$.

Finally, the effect of varying the harvesting rate of infected predator , in the range $0.04 < c_{14} < 0.188$ keeping other parameters as data given in (6.1) is studied, it is observed that system (2.2) still approach asymptotically to the positive equilibrium point, however $0.188 \le c_{14} \le 0.5$ causes extinction in the infected prey and the system will approach the infected prey free equilibrium point.

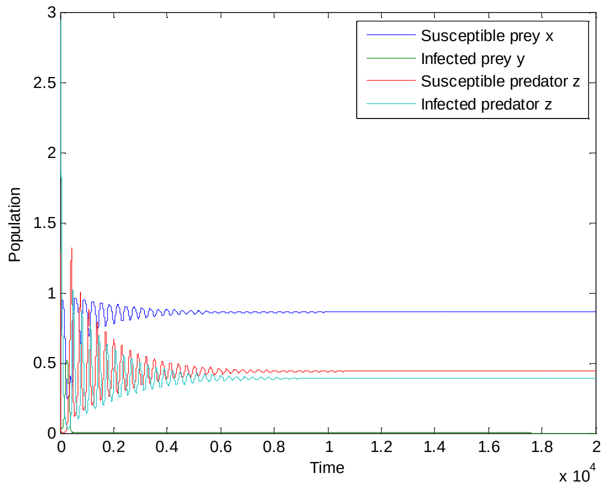

Fig 8 Time series of the solution of system (2.2) approaches asymptotically to the infected prey free equilibrium point E_5 = (0.86, 0,0.43 , 0.38) for the data given in (6.1) with c_{14} = 0.2.

CONCLUSIONS AND DISCUSSIONS

In the previous sections, prey-predator model, with SI epidemic disease in both species and harvest in the infected population, is proposed and analyzed. It is assumed that the disease is transmitted within the individuals through contact , the uniqueness and boundedness of solution of the system are discussed, the existence of all possible equilibrium points are investigated, it is observed that system(2.2) has at most seven nonnegative equilibrium points in R^4_+ . The dynamical behavior of system (2.2) has been investigated locally as well as globally. Further, it is observed that the vanishing equilibrium point E_0 always exist, and it is unstable. The axial equilibrium point E_1 always exist, and it is locally asymptotically stable point if and only if conditions (4.3) and (4.4) hold, in addition to that it is globally if the condition (5.1). The predator free equilibrium point E_2 exist under the condition (3.1), it is locally asymptotically stable point if and only if conditions (4.7) and (4.8) hold, as well as it is globally if the conditions (5.2) hold. The disease free equilibrium point E_3 exist under the conditions (3.2) and (3.2), it is locally asymptotically stable point if and only if conditions (4.11) and (4.12) hold, as well as it is globally if the conditions (5.3) –(5.5) hold. The infected predator free equilibrium point E_4 exist under the conditions (3.10)-(3.12), it is locally asymptotically stable point if and only if conditions (4.16) and (4.18)-(4.20) hold, as well as it is globally if the conditions (5.6)-(5.9) hold The infected pry free equilibrium point E_5 exist under the conditions (3.18) and (3.19), it is locally asymptotically stable point if and only if conditions (4.24) and (4.25) hold, as well as it is globally if the conditions (5.10)-(5.13) hold. The positive equilibrium point E_6 of system (2.2) exist provided that the conditions (3.31) *and* (3.34) are hold and the isocline $g_1(x, y)$ = 0 intersect the x-axis at the positive value namelyx_1^*. It islocally asymptotically stable point if and only if conditions (4.29)-(4.34) hold, in addition it is globally if the conditions (5.14)-(5.17) hold.

To understand the effect of varying each parameter including harvest on the global dynamics of system (2.2) and to confirm our above analytical results, system (2.2) has been solved numerically and the following results are obtained:

1. For the set of hypothetical parameters values given in $(\mathbf{6.1})$, system $(\mathbf{2.2})$ approaches asymptotically to a globally asymptotically stable point $\boldsymbol{E_6}$ = (0.688 ,0.164 ,0.174 ,0.491) .
2. Varying the conversion rate parameter value and the half saturation parameter c_8 , c_9, c_{12} and c_3 respectively at each time keeping other parameters fixed as data given in (6.1) do not have any effect on the dynamical behavior of system (2.2) and the solution of the system still approaches to positive equilibrium point E_6 = (x^* , y^* , z^* , w^*) .
3. Increasing the infection parameter c_1 > 0.37 and c_{10} > 1.5 at each time keeping other parameters fixed as data given in (6.1) the solution of the system (2.2) will approaches asymptotically to the vanishing equilibrium point. E_6 = (x^* , y^* , z^* , w^*)
4. Further, Increasing the maximum attack rate of susceptible predator for susceptible prey ,harvesting rate and death of infected predator rare due to disease parameter c_2 > 1.45, 0.215 < c_7 , c_{13} ≥ 1.5 , and c_{14} ≥ 0.89 respectively at each time keeping other parameters fixed as data given in (6.1) the solution of the system (2.2) will approaches asymptotically to the infected prey free equilibrium point E_5 = $(\tilde{x}, 0, \tilde{z}, \widetilde{w})$
5. Varying the conversion rate of the susceptible prey from susceptible predator and infection rate of prey, c_8 < 0.012 and c_1 < 0.1 keeping other parameters as data given in (6.1) , it is makes the solution of system (2.2) approaches asymptotically to the axial equilibrium point E_1 = (1,0, 0,0)
6. Increasing the maximum attack rate of susceptible predator for infected prey, the maximum attack rate of infected predator for infected prey, death of infected prey rare due to disease 0.98 < c_4 < 1.5 ,0.59 < c_5 < 1.5 ,0.124 < c_6 < 1 respectively at each time keeping other parameters fixed as data given in (6.1) the solution of the system (2.2) will approaches asymptotically to the infected prey free equilibrium point E_5 = $(\tilde{x}, 0, \tilde{z}, \widetilde{w})$
7. Finally, Varying the death rate parameter value of predator c_{11} = 0.2 the solution of the system (2.2) approaches to the disease free equilibrium point$\boldsymbol{E_3}$ = $(\dot{\boldsymbol{x}}, \mathbf{0}, \dot{\boldsymbol{z}}, \mathbf{0})$, while 0.2 < c_{11} ≤ 0.31 causes extinction in the infected predator and

the system will approach the infected predator free equilibrium point$E_4 = (\bar{\bar{x}}, \bar{\bar{y}}, \bar{\bar{z}}, 0)$. Now, increase $0.31 < c_{11}$ the solution of the system (2.2) approaches to the predator free equilibrium point.

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