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Research Article

# NEW APPROACH OF FIXED POINT THEOREM IN A COMPLETE METRIC SPACE

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**ABSTRACT**

In this paper some fixed point theorems have been proved in a complete metric space which generalized the classical Banach contraction mapping principle and many results of great mathematicians.



## INTRODUCTION

The Polish mathematician Stefan Banach[1] proved a theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. It is well known as a Banach fixed point theorem. The existence of a fixed point plays an important role in several areas of mathematics, physics and engineering branches. This principle has been generalized by many authors in various ways.

Kanan[8] proved that, If T is a self mapping from a complete metric space X into itself with $d(Tx,Ty) \leq \alpha[d(Tx, x) + d(Ty, y)$ for all x, y ϵ X, where α ϵ [0, ½], then T has a unique fixed point in X.

Reich[3] proved this result with $d(Tx,Ty) \leq \alpha[d(Tx, x) + d(Ty, y) + \beta[d(Ty, x) + d(Tx, y)] + \gamma d(x, y)$, for all x, y ϵ X, where α ϵ [0, ½].

Fisher[7] in the same way proved this result with $d(Tx,Ty) \leq \alpha[d(Ty, x) + d(Tx, y)$ for all x, y ϵ X, where
α ϵ [0, ½].

After that Chaterjee[6] proved that the same result for $d(Tx,Ty) \leq \alpha[d(Tx, x) + d(Ty, y) + \beta d(x, y)$ for all x, y ϵ X, where α ϵ [0, ½].

The aim of this paper is to obtain a fixed point theorem for new rational inequality in complete metric space which satisfies the many results of great mathematicians.

***Main results***

***Theorem:*** Let f be a continuous self mapping defined on complete metric space (X, d) such that

$$d(fx,fy) \leq \alpha \frac{d(x,fx).d(y,fy) + d(x,fx)d(y,fx)}{d(x,y)}$$
$$+ \beta \frac{d(x,fx)d(y,fx) + d(y,fy)d(x,fy)}{d(x,fx) + d(y,fx) + d(y,fy) + d(x,fy)} + \gamma \frac{d(x,fy)[d(x,fx) + d(y,fy)]}{d(x,y) + d(y,fy) + d(y,fx)} +$$

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$\xi \dfrac{d(x,fx)[d(x,fy)+d(y,fx)]}{d(x,y)+d(y,fy)+d(y,fx)} + \delta[d(x, fx) + d(y, fy)] + \eta[d(y, fx) + d(x, fy)] +$
$\mu d(x, y)$ (1)

For all x, y $\epsilon$ X, $x \neq y$ and $\alpha, \beta, \gamma, \delta, \eta, \mu \in [0, 1)$ with $2\alpha + 2\beta + \gamma + 4\delta + 4\eta + 2\mu < 2$. Then f has a unique fixed point in T.

***Proof**:* Define a sequence $\{x_n\}$ by setting $T^n x_0 = x_n$, where n is a positive integer. Taking $x_n \neq x_{n+1,}$ then by (1)

$$d(x_{n+1}, x_n) = d(fx_n, fx_{n-1}) \leq \alpha \frac{d(x_n,fx_n).d(x_{n-1},fx_{n-1})+d(x_n,fx_{n-1})d(x_{n-1},fx_n)}{d(x_n,x_{n-1})} + \beta$$

$$\frac{d(x_n,fx_n).d(x_{n-1},fx_n)+d(x_{n-1},fx_{n-1})d(x_{n,}fx_{n-1}]}{d(x_n,fx_n)+d(x_{n-1},fx_n)+d(x_{n-1,}fx_{n-1})+d(x_{n,}fx_{n-1})}$$

$$+\gamma \frac{d(x_{n-1},fx_{n-1})[.d(x_{n-1},fx_{n-1})+d(x_n,fx_n)]}{d(x_n,x_{n-1})+d(x_{n-1},fx_{n-1})+d(x_{n-1,}fx_n)}$$

$$+ \xi \frac{d(x_n,fx_n)[.d(x_n,fx_{n-1})+d(x_{n-1},fx_n)]}{d(x_n,x_{n-1})+d(x_{n-1},fx_{n-1})+d(x_{n-1},fx_n)} + \delta[d(x_n, fx_n) + d(x_{n-1}, fx_{n-1})] +$$

$\eta[d(x_{n-1}, fx_n) + d(x_n, fx_{n-1})] + \mu d(x_n, x_{n-1})$
or $d(fx_n, fx_{n-1}) \leq (\alpha + \beta/2 + + \delta + \eta)d(x_n, x_{n+1}) + (\delta + \eta + \mu)d(x_{n-1}, x_n)$

i.e. $d(x_{n+1}, x_n) \leq \dfrac{\delta+\eta+\mu}{1-(\alpha+\beta+\gamma/2+\delta+\eta)} d(x_{n-1}, x_n)$
$= \lambda\, d(x_{n-1}, x_n)$

Where $\lambda = \dfrac{\delta+\eta+\mu}{1-(\alpha+\beta+\gamma/2+\delta+\eta)}$ with $0 \leq \lambda < 1$.

In a similar way we can show that $d(x_{n+1}, x_n) \leq \lambda^n d(x_0, x_1)$.

By triangle inequality we have for $m \geq n$,
$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ......... + d(x_{m-1}, x_m)$
$\leq (\lambda^n + \lambda^{n+1} + ....... + \lambda^{m-1})\, d(x_{0,}\ x_1)$
$\leq \dfrac{\lambda^n}{1-\lambda} d(x_{0,} x_1)$

Since $0 \leq \lambda < 1$,as $n \to \infty$, $\lambda^n \to 0$ which implies that $d(x_n, x_m) \to 0$ i.e. $\{x_n\}$ is a cauchy sequence.

So by completeness of X this sequence must be converge to u i. e. $\{x_n\} \to x$ as $n \to \infty$. Further, continuity of T in X implies $T(x) = T(\lim_{n\to\infty}x_n) = \lim_{n\to\infty}Tx_n = \lim_{n\to\infty}x_{n+1} = x$. Therefore x is a fixed point of T.

***Uniqueness:*** Let $y \neq x$ be another fixed point of f, where f(y) = y. Then by given condition, we have
$d(x, y) = d(f(x), f(y))$

$$\leq \alpha \frac{d(x,fx).d(y,fy)+d(x,fx)d(y,fx)}{d(x,y)} + \beta \frac{d(x,fx)[d(x,fy)+d(y,fx)]}{d(x,y)+d(y,fy)+d(y,fx)} + \gamma \frac{d(x,fx)d(y,fx)+d(y,fy)d(x,fy)}{d(x,fx)+d(y,fx)+d(y,fy)+d(x,fy)} +$$
$\delta[d(x, fx) + d(y, fy)] + \eta[d(y, fx) + d(x, fy)] +$

$\mu d(x, y)$
i.e. $d(x, y) \leq (\alpha + 2\eta + \mu)\, d(x, y)$.

Since $2\alpha + 2\beta + \gamma + 4\delta + 4\eta + 2\mu < 2$, we obtained d(x, y) = 0, which implies x = y. Thus x is a unique fixed point of f.

***Theorem**:* Let f be a self mapping defined on complete metric space (X, d) such that (1) holds. If for some positive integer m, $f^m$ is continuous then f has a unique fixed point.

***Proof**:* Define a sequence $\{x_n\}$ by setting $f^n x_0 = x_n$, where n is a positive integer. Then $\{x_n\}$ converges to some point x in X. So the subsequence $\{x_{nk}\}$ of $\{x_n\}$ is also converges to x.

So $f_x^m = f^m(\lim_{k\to\infty}fx_{nk}) = (\lim_{k\to\infty}f^m x_{nk}) = (\lim_{k\to\infty}x_{nk+m}) = x$
Therefore x is a fixed point of $f_x$.

Now consider that p be the smallest positive integer such that $f_x^P = x$ but $f_x^q \neq x$ for q = 1,2,3......p-1. If p > 1, then

$d(f_x, x) = d(f_x, f_x^p) = d(f_x, f(f_x^{p-1}))$

$$\leq \frac{d(x,fx).d(f^{m-1}x,f^mx)+d(x,f^mx)d(f^{m-1}x,fx)}{d(x,f^{m-1}x)}+\beta\frac{d(x,fx)[d(x,f^px)+d(f^{p-1}x,fx)]}{d(x,f^{p-1}x)+d(f^{p-1}x,f^px)+d(f^{p-1}x,fx)}+\gamma$$

$$\frac{d(x,fx)d(f^{p-1}x,fx)+d(f^{p-1}x,fx)d(x,f^mx)}{d(x,fx)+d(f^{p-1}x,fx)+d(f^{p-1}x,f^mx)+d(x,f^mx)}+\delta[d(x,fx)+d(f^{p-1}x,f^px)]+$$

$\eta[d(f^{p-1}x, fx) + d(x, f^px)] + \mu d(x, f^{p-1}x)$

i.e. $d(x, f_x) \leq \dfrac{\delta+\eta+\mu}{1-(\alpha+\gamma/2+\delta+\eta)}\ d(x, f^{p-1}x)$

or $d(x, f_x) \leq \lambda\ d(x, f^{p-1}x)$, where $\lambda = \dfrac{\delta+\eta+\mu}{1-(\alpha+\gamma/2+\delta+\eta)}$

Thus we can write, $d(x, fx) \leq \lambda^p\ d(x, fx)$

But $\lambda^p < 1$, we get a contradiction. Thus $T_x = x$ i.e. x is a fixed point of f.
Uniqueness follows as in theorem 1.

***Theorem:*** Let f be a continuous self mapping defined on complete metric space (X, d) such that for some positive integer p, f satisfies:

$$d(f^px, f^py) \leq \alpha\frac{d(x,f^px).d(y,f^py)+d(x,f^px)d(y,f^px)}{d(x,y)}+\beta\frac{d(x,f^px)[d(x,f^py)+d(y,f^px)]}{d(x,y)+d(y,f^py)+d(y,f^px)}+\gamma$$

$$\frac{d(x,f^px)d(y,f^px)+d(y,f^py)d(x,f^py)}{d(x,f^px)+d(y,f^px)+d(y,f^py)+d(x,f^py)}+$$

$\delta[d(x, f^px) + d(y, f^py)] + \eta[d(y, f^px) + d(x, f^py)] + \mu d(x, y)$

For all $x, y \in X$, $x \neq y$ and $\alpha, \beta, \gamma, \delta, \eta, \mu \in [0, 1)$ with $2\alpha + 2\beta + \gamma + 4\delta + 4\eta + 2\mu < 2$. If $f^p$ is continuous then f has a unique fixed point.

***Proof***: By theorem 2, $f^p$ has a fixed point with $fx = f(f^px) = f^p(fx)$ so we get $fx = x$. Again fixed point of f is a fixed point of $f^p$ and $f^p$ has fixed point x, so x is the unique fixed point of f.

***Example***: Let X = [0, 1] with the usual metric and $f : X \to X$ defined by
$fx = \{ 0$, when $0 \leq x \leq 1/3$
$= \{1/3$, when $1/3 < x \leq 1$.

Obviously f is discontinuous and does not satisfy theorem 1 when x = 1/3 and y = 1. But clearly $f^2$ is continuous and satisfy theorem 3 with 0 is the unique fixed point of $f^2$ and so of f.

***Remark***

1. If we put $\alpha = \beta = \gamma = \delta = \eta = 0$ we obtained the result of Banach [1].
2. If we put $\alpha = \beta = \gamma = \eta = \mu = 0$ we obtained the result of Kannan [8].
3. If we put $\alpha = \beta = \gamma = 0$ we obtained the result of Reich [3].
4. If we put $\alpha = \beta = \gamma = \eta = 0$ we obtained the result of Chatterjee [6].
5. If we put $\alpha = \beta = \gamma = \delta = 0$ we obtained the result of Fisher [7].

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