International Journal of Current Advanced Research
ISSN: O: 2319-6475, ISSN: P: 2319 – 6505, Impact Factor: SJIF: 5.438
Available Online at www.journalijcar.org
Volume 6; Issue 2; February 2017; Page No. 2081-2086

Research Article

NUMBER OF ZEROS OF A POLYNOMIAL IN A CLOSED DISC

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A R T I C L E I N F O

Article History:

Received 29th November, 2016
Received in revised form 30thDecember, 2016
Accepted 4th January, 2017
Published online 28th February, 2017

Key words:

Coefficients, Polynomial, Zeros.

A B S T R A C T

In this paper we find the number of zeros of a polynomial in a closed disc, when the real and imaginary parts of the coefficients of the polynomial are restricted to certain conditions.

INTRODUCTION

In connection with a generalization of the Enestom-Kakeya Theorem [3,4] which states that all the n zeros of an nth degree polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ with $a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0$ lie in $|z| \le 1$, Gulzar [2] very recently proved the following result:

Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j$, $\mathrm{Im}(a_j)=\beta_j$, $j=0,1,2,......,n$ such that for some $\lambda,\mu; 0 \le \lambda \le n-1, 0 \le \mu \le n-1$ and for some $k_1,k_2 \le 1; \tau_1,\tau_2 \ge 1$,

$$k_1\alpha_n \le \alpha_{n-1} \le ...... \le \tau_1\alpha_\lambda$$
$$k_2\beta_n \le \beta_{n-1} \le ...... \le \tau_2\beta_\mu,$$

and

$$L = |\alpha_\lambda - \alpha_{\lambda-1}| + |\alpha_{\lambda-1} - \alpha_{\lambda-2}| + ... + |\alpha_1 - \alpha_0| + |\alpha_0|,$$
$$M = |\beta_\mu - \beta_{\mu-1}| + |\beta_{\mu-1} - \beta_{\mu-2}| + ... + |\beta_1 - \beta_0| + |\beta_0|,$$

Then all the zeros of P(z) lie in

$$\left| z - \frac{(1-k_1)\alpha_n + i(1-k_2)\beta_n}{a_n} \right| \le \frac{1}{|a_n|}[\tau_1(\alpha_\lambda + |\alpha_\lambda|) + \tau_2(\beta_\mu + |\beta_\mu|) - |\alpha_\lambda| - |\beta_\mu| - k_1\alpha_n - k_2\beta_n + L + M].$$

2. Main RESULTS

In this paper we prove the following result:

Theorem 1: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j$, $\mathrm{Im}(a_j)=\beta_j$, $j=0,1,2,......,n$ such that for some $\lambda,\mu; 0 \le \lambda \le n-1, 0 \le \mu \le n-1$ and for some $k_1,k_2 \le 1; \tau_1,\tau_2 \ge 1$,

$$k_1\alpha_n \le \alpha_{n-1} \le ...... \le \tau_1\alpha_\lambda$$
$$k_2\beta_n \le \beta_{n-1} \le ...... \le \tau_2\beta_\mu,$$

and

$$L = |\alpha_\lambda - \alpha_{\lambda-1}| + |\alpha_{\lambda-1} - \alpha_{\lambda-2}| + ... + |\alpha_1 - \alpha_0| + |\alpha_0|,$$
$$M = |\beta_\mu - \beta_{\mu-1}| + |\beta_{\mu-1} - \beta_{\mu-2}| + ... + |\beta_1 - \beta_0| + |\beta_0|,$$

Then the number of zero of P(z) in $\frac{|a_0|}{X} \le |z| \le \frac{R}{c}, c > 1$ is less than or equal to $\frac{1}{\log c}\log\frac{A}{|a_0|}$ for $R \ge 1$ and the number of zeros of P(z) in $\frac{|a_0|}{Y} \le |z| \le \frac{R}{c}, c > 1$ is less than or equal to $\frac{1}{\log c}\log\frac{B}{|a_0|}$ for $R \le 1$, where

$$X = |a_n|R^{n+1} + R^n[|\alpha_n| + |\beta_n| - k_{1(}(|\alpha_n| + \alpha_n) - k_2(|\beta_n| + \beta_n) + \tau_1(|\alpha_\lambda| + \alpha_\lambda)$$
$$+ \tau_2(|\beta_\mu| + \beta_\mu) - |\alpha_\lambda| - |\beta_\mu| + L + M - |\alpha_0| - |\beta_0|\},$$
$$Y = |a_n|R^{n+1} + R[|\alpha_n| + |\beta_n| - k_{1(}(|\alpha_n| + \alpha_n) - k_2(|\beta_n| + \beta_n) + \tau_1(|\alpha_\lambda| + \alpha_\lambda)$$
$$+ \tau_2(|\beta_\mu| + \beta_\mu) - |\alpha_\lambda| - |\beta_\mu| + L + M] - (1-R)(|\alpha_0| + |\beta_0|),$$
$$A = |a_n|R^{n+1} + R^n[|\alpha_n| + |\beta_n| - k_{1(}(|\alpha_n| + \alpha_n) - k_2(|\beta_n| + \beta_n) + \tau_1(|\alpha_\lambda| + \alpha_\lambda)$$
$$+ \tau_2(|\beta_\mu| + \beta_\mu) - |\alpha_\lambda| - |\beta_\mu| + L + M],$$
$$B = |a_n|R^{n+1} + R[|\alpha_n| + |\beta_n| - k_{1(}(|\alpha_n| + \alpha_n) - k_2(|\beta_n| + \beta_n) + \tau_1(|\alpha_\lambda| + \alpha_\lambda)$$
$$+ \tau_2(|\beta_\mu| + \beta_\mu) - |\alpha_\lambda| - |\beta_\mu| + L + M] - (1-R)(|\alpha_0| + |\beta_0|).$$

For particular values of the parameters, we get many interesting results from Theorem 1. For example, for R=1, $c = \delta, 0 < \delta < 1$, we get the following result:

Corollary 1: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$, $\text{Im}(a_j) = \beta_j$, $j = 0,1,2,......,n$ such that for some $\lambda, \mu; 0 \le \lambda \le n-1, 0 \le \mu \le n-1$ and for some $k_1, k_2 \le 1; \tau_1, \tau_2 \ge 1$,

$$k_1\alpha_n \le \alpha_{n-1} \le ...... \le \tau_1\alpha_\lambda$$
$$k_2\beta_n \le \beta_{n-1} \le ...... \le \tau_2\beta_\mu,$$

and

$$L = |\alpha_\lambda - \alpha_{\lambda-1}| + |\alpha_{\lambda-1} - \alpha_{\lambda-2}| + ... + |\alpha_1 - \alpha_0| + |\alpha_0|,$$
$$M = |\beta_\mu - \beta_{\mu-1}| + |\beta_{\mu-1} - \beta_{\mu-2}| + ... + |\beta_1 - \beta_0| + |\beta_0|,$$

Then the number of zero of P(z) in $\frac{|a_0|}{X} \le |z| \le \delta, 0 < \delta < 1$ is less than or equal to $\frac{1}{\log\frac{1}{\delta}}\log\frac{A}{|a_0|}$, where

$$X = |a_n| + |\alpha_n| + |\beta_n| - k_{1(}(|\alpha_n| + \alpha_n) - k_2(|\beta_n| + \beta_n) + \tau_1(|\alpha_\lambda| + \alpha_\lambda)$$
$$+ \tau_2(|\beta_\mu| + \beta_\mu) - |\alpha_\lambda| - |\beta_\mu| + L + M - |\alpha_0| - |\beta_0|,$$
$$A = |a_n| + |\alpha_n| + |\beta_n| - k_{1(}(|\alpha_n| + \alpha_n) - k_2(|\beta_n| + \beta_n) + \tau_1(|\alpha_\lambda| + \alpha_\lambda)$$
$$+ \tau_2(|\beta_\mu| + \beta_\mu) - |\alpha_\lambda| - |\beta_\mu| + L + M.$$

For $\tau_1 = \tau_2 = 1$, we get the following result from Theorem 1:

Corollary 2: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$, $\text{Im}(a_j) = \beta_j$, $j = 0,1,2,......,n$ such that for some $\lambda, \mu; 0 \le \lambda \le n-1, 0 \le \mu \le n-1$ and for some $k_1, k_2 \le 1$,

$$k_1\alpha_n \le \alpha_{n-1} \le ......\le \alpha_\lambda$$
$$k_2\beta_n \le \beta_{n-1} \le ......\le \beta_\mu,$$

and

$$L = |\alpha_\lambda - \alpha_{\lambda-1}| + |\alpha_{\lambda-1} - \alpha_{\lambda-2}| + ...+ |\alpha_1 - \alpha_0| + |\alpha_0|,$$
$$M = |\beta_\mu - \beta_{\mu-1}| + |\beta_{\mu-1} - \beta_{\mu-2}| + ...+ |\beta_1 - \beta_0| + |\beta_0|,$$

Then the number of zero of P(z) in $\frac{|a_0|}{X} \le |z| \le \frac{R}{c}, c > 1$ is less than or equal to $\frac{1}{\log c}\log\frac{A}{|a_0|}$ for $R \ge 1$ and the number of zeros of P(z) in $\frac{|a_0|}{Y} \le |z| \le \frac{R}{c}, c > 1$ is less than or equal to $\frac{1}{\log c}\log\frac{B}{|a_0|}$ for $R \le 1$, where

$$X = |a_n|R^{n+1} + R^n[|\alpha_n| + |\beta_n| - k_{1(}(|\alpha_n| + \alpha_n) - k_2(|\beta_n| + \beta_n) - \alpha_\lambda$$
$$- \beta_\mu + L + M - |\alpha_0| - |\beta_0|\},$$
$$Y = |a_n|R^{n+1} + R[|\alpha_n| + |\beta_n| - k_{1(}(|\alpha_n| + \alpha_n) - k_2(|\beta_n| + \beta_n) - \alpha_\lambda$$
$$- \beta_\mu + L + M] - (1-R)(|\alpha_0| + |\beta_0|),$$
$$A = |a_n|R^{n+1} + R^n[|\alpha_n| + |\beta_n| - k_{1(}(|\alpha_n| + \alpha_n) - k_2(|\beta_n| + \beta_n) - \alpha_\lambda$$
$$- \beta_\mu + L + M],$$
$$B = |a_n|R^{n+1} + R[|\alpha_n| + |\beta_n| - k_{1(}(|\alpha_n| + \alpha_n) - k_2(|\beta_n| + \beta_n) - \alpha_\lambda$$
$$- \beta_\mu + L + M] - (1-R)(|\alpha_0| + |\beta_0|).$$

For $k_1 = k_2 = \tau_1 = \tau_2 = 1$, we get the following result from Theorem 1:

Corollary 3: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j$, $\text{Im}(a_j) = \beta_j$, $j = 0,1,2,......,n$ such that for some $\lambda, \mu; 0 \le \lambda \le n-1, 0 \le \mu \le n-1$,

$$\alpha_n \le \alpha_{n-1} \le ......\le \alpha_\lambda$$
$$\beta_n \le \beta_{n-1} \le ......\le \beta_\mu,$$

and

$$L = |\alpha_\lambda - \alpha_{\lambda-1}| + |\alpha_{\lambda-1} - \alpha_{\lambda-2}| + ...+ |\alpha_1 - \alpha_0| + |\alpha_0|,$$
$$M = |\beta_\mu - \beta_{\mu-1}| + |\beta_{\mu-1} - \beta_{\mu-2}| + ...+ |\beta_1 - \beta_0| + |\beta_0|,$$

Then the number of zero of P(z) in $\frac{|a_0|}{X} \le |z| \le \frac{R}{c}, c > 1$ is less than or equal to $\frac{1}{\log c}\log\frac{A}{|a_0|}$ for $R \ge 1$ and the number of zeros of P(z) in $\frac{|a_0|}{Y} \le |z| \le \frac{R}{c}, c > 1$ is less than or equal to $\frac{1}{\log c}\log\frac{B}{|a_0|}$ for $R \le 1$, where

$$X = |a_n|R^{n+1} + R^n[\alpha_n + \beta_n - \alpha_\lambda - \beta_\mu + L + M - |\alpha_0| - |\beta_0|],$$
$$Y = |a_n|R^{n+1} + R[\alpha_n) + \beta_n - \alpha_\lambda - \beta_\mu + L + M] - (1-R)(|\alpha_0| + |\beta_0|),$$
$$A = |a_n|R^{n+1} + R^n[-\alpha_n - \beta_n - \alpha_\lambda - \beta_\mu + L + M],$$
$$B = |a_n|R^{n+1} + R[-\alpha_n - \beta_n - \alpha_\lambda - \beta_\mu + L + M] - (1-R)(|\alpha_0| + |\beta_0|).$$

Lemmas

For the proof of Theorem 2, we make use of the following lemmas:

Lemma 1: Let f(z) (not identically zero) be analytic for $|z| \le R, f(0) \ne 0$ and $f(a_k) = 0, \quad k = 1,2,......,n$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(\text{Re}^{i\theta}|d\theta - \log|f(0)| = \sum_{j=1}^{n} \log\frac{R}{|a_j|}.$$

Lemma 1 is the famous Jensen's Theorem (see page 208 of [1]).

Lemma 2: Let f(z) be analytic , $f(0) \neq 0$ and $|f(z)| \leq M$ for $|z| \leq R$. Then the number of zeros of f(z) in $|z| \leq \frac{R}{c}, c > 1$ is less than or equal to $\frac{1}{\log c} \log \frac{M}{|f(0)|}$.

Lemma 2 is a simple deduction from Lemma 1.

4. Proofs of Theorems

Proof of Theorem 1: Consider the polynomial

$$\begin{aligned}
F(z) &= (1-z)P(z) \\
&= (1-z)(a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0) \\
&= -a_n z^{n+1} + (a_n - a_{n-1}) z^n + ... + (a_{\lambda+1} - a_\lambda) z^{\lambda+1} + (a_\lambda - a_{\lambda-1}) z^\lambda \\
&\quad + ... + (a_1 - a_0) z + a_0 \\
&= -a_n z^{n+1} - (k_1 - 1)\alpha_n z^n + (k_1 \alpha_n - \alpha_{n-1}) z^n + (\alpha_{n-1} - \alpha_{n-2}) z^{n-1} ...... + (\alpha_{\lambda+1} - \tau_1 \alpha_\lambda) z^{\lambda+1} \\
&\quad + (\tau_1 - 1)\alpha_\lambda z^{\lambda+1} + (\alpha_\lambda - \alpha_{\lambda-1}) z^\lambda + ... + (\alpha_1 - \alpha_0) z + i\{(k_2 \beta_n - \beta_{n-1}) z^n \\
&\quad - (k_2 - 1)\beta_n z^n + ... + (\beta_{\mu+1} - \tau_2 \beta_\mu) z^{\mu+1} + (\tau_2 - 1)\beta_\mu z^{\mu+1} + (\beta_\mu - \beta_{\mu-1}) z^\mu \\
&\quad + ... + (\beta_1 - \beta_0) z\} + a_0 \\
&= G(z) + a_0,
\end{aligned}$$

where

$$\begin{aligned}
G(z) &= -a_n z^{n+1} - (k_1 - 1)\alpha_n z^n + (k_1 \alpha_n - \alpha_{n-1}) z^n + (\alpha_{n-1} - \alpha_{n-2}) z^{n-1} ...... + (\alpha_{\lambda+1} - \tau_1 \alpha_\lambda) z^{\lambda+1} \\
&\quad + (\tau_1 - 1)\alpha_\lambda z^{\lambda+1} + (\alpha_\lambda - \alpha_{\lambda-1}) z^\lambda + ... + (\alpha_1 - \alpha_0) z + i\{(k_2 \beta_n - \beta_{n-1}) z^n \\
&\quad - (k_2 - 1)\beta_n z^n + ... + (\beta_{\mu+1} - \tau_2 \beta_\mu) z^{\mu+1} + (\tau_2 - 1)\beta_\mu z^{\mu+1} + (\beta_\mu - \beta_{\mu-1}) z^\mu \\
&\quad + ... + (\beta_1 - \beta_0) z\}
\end{aligned}$$

For $|z| = R$, we have, by using the hypothesis

$$\begin{aligned}
|G(z)| &\leq |a_n||z|^{n+1} + (1-k_1)|\alpha_n||z|^n + (1-k_2)|\beta_n||z|^n + [|k_1\alpha_n - \alpha_{n-1}||z|^n + |\alpha_{n-1} - \alpha_{n-2}||z|^{n-1} + ... \\
&\quad + |\alpha_{\lambda+1} - \tau_1\alpha_\lambda||z|^{\lambda+1} + (\tau_1 - 1)|\alpha_\lambda||z|^{\lambda+1} + |\alpha_\lambda - \alpha_{\lambda-1}||z|^\lambda + ... + |\alpha_1 - \alpha_0||z| \\
&\quad + |k_2\beta_n - \beta_{n-1}||z|^n + |\beta_{n-1} - \beta_{n-2}||z|^{n-1} ...... + |\beta_{\mu+1} - \tau_2\beta_\mu||z|^{\mu+1} + (\tau_2 - 1)|\beta_\mu||z|^\mu \\
&\qquad + |\beta_\mu - \beta_{\mu-1}||z|^{\mu-1} + ... + |\beta_1 - \beta_0||z|] \\
&= |a_n|R^{n+1} + (1-k_1)|\alpha_n|R^n + (1-k_2)|\beta_n|R^n - [|k_1\alpha_n - \alpha_{n-1}|R^n + |\alpha_{n-1} - \alpha_{n-2}|R^{n-1} + ... \\
&\quad + |\alpha_{\lambda+1} - \tau_1\alpha_\lambda|R^{\lambda+1} + (\tau_1 - 1)|\alpha_\lambda|R^{\lambda+1} + |\alpha_\lambda - \alpha_{\lambda-1}|R^\lambda + ... + |\alpha_1 - \alpha_0| \\
&\quad + |k_2\beta_n - \beta_{n-1}|R^n + |\beta_{n-1} - \beta_{n-2}|R^{n-1} + ... + |\beta_{\mu+1} - \tau_2\beta_\mu|R^{\mu+1} + (\tau_2 - 1)|\beta_\mu|R^{\mu+1} \\
&\quad + ... + |\beta_1 - \beta_0|R\}] \\
&\leq |a_n|R^{n+1} + R^n[(1-k_1)|\alpha_n| + (1-k_2)|\beta_n| + |k_1\alpha_n - \alpha_{n-1}| + |\alpha_{n-1} - \alpha_{n-2}| + ... \\
&\quad + |\alpha_{\lambda+1} - \tau_1\alpha_\lambda| + (\tau_1 - 1)|\alpha_\lambda| + |\alpha_\lambda - \alpha_{\lambda-1}| + ... + |\alpha_1 - \alpha_0| \\
&\quad + |k_2\beta_n - \beta_{n-1}| + |\beta_{n-1} - \beta_{n-2}| + ... + |\beta_{\mu+1} - \tau_2\beta_\mu| + (\tau_2 - 1)|\beta_\mu| \\
&\quad + ... + |\beta_1 - \beta_0|] \\
&= |a_n|R^{n+1} + R^n[(1-k_1)|\alpha_n| + (1-k_2)|\beta_n| + \alpha_{n-1} - k_1\alpha_n + \alpha_{n-2} - \alpha_{n-1} + ... \\
&\quad + \tau_1\alpha_\lambda - \alpha_{\lambda+1} + (\tau_1 - 1)|\alpha_\lambda| + |\alpha_\lambda - \alpha_{\lambda-1}| + ... + |\alpha_1 - \alpha_0|
\end{aligned}$$

$$+\beta_{n-1}-k_2\beta_n+\beta_{n-2}-\beta_{n-1}+......+\tau_2\beta_\mu-\beta_{\mu+1}+(\tau_2-1)|\beta_\mu|$$
$$+......+|\beta_1-\beta_0|]$$
$$=|a_n|R^{n+1}+R^n[|\alpha_n|+|\beta_n|-k_{1(}(|\alpha_n|+\alpha_n)-k_2(|\beta_n|+\beta_n)+\tau_1(|\alpha_\lambda|+\alpha_\lambda)$$
$$+\tau_2(|\beta_\mu|+\beta_\mu)-|\alpha_\lambda|-|\beta_\mu|+L+M-|\alpha_0|-|\beta_0|\}$$
=X

for $R\geq 1$

and for $R\leq 1$

$$|G(z)|\leq|a_n|R^{n+1}+R[|\alpha_n|+|\beta_n|-k_{1(}(|\alpha_n|+\alpha_n)-k_2(|\beta_n|+\beta_n)+\tau_1(|\alpha_\lambda|+\alpha_\lambda)$$
$$+\tau_2(|\beta_\mu|+\beta_\mu)-|\alpha_\lambda|-|\beta_\mu|+L+M]-(1-R)(|\alpha_0|+|\beta_0|)$$
=Y

Since G (z) is analytic for $|z|\leq R, G(0)=0$, it follows by Schwarz Lemma that

$|G(z)|\leq X|z|$ for $R\geq 1$ and $|G(z)|\leq Y|z|$ for $R\leq 1$.

Hence for $R\geq 1$,

$$|F(z)|=|a_0+G(z)|$$
$$\geq|a_0|-|G(z)|$$
$$\geq|a_0|-X|z|$$
$$>.0$$

if $|z|<\dfrac{|a_0|}{X}$

and for $R\leq 1$

$$|F(z)|>0$$

if $|z|<\dfrac{|a_0|}{Y}$.

This shows that F(z) and hence P(z) does not vanish in $|z|<\dfrac{|a_0|}{X}$ for $R\geq 1$ and in $|z|<\dfrac{|a_0|}{Y}$ for $R\leq 1$. In other words all the zeros of P(z) lie in $|z|\geq\dfrac{|a_0|}{X}$ for $R\geq 1$ and in $|z|\geq\dfrac{|a_0|}{Y}$ for $R\leq 1$.

Again , for $|z|\leq R$, it is easy to see as above that

$$|F(z)|\leq|a_n|R^{n+1}+R^n[|\alpha_n|+|\beta_n|-k_{1(}(|\alpha_n|+\alpha_n)-k_2(|\beta_n|+\beta_n)+\tau_1(|\alpha_\lambda|+\alpha_\lambda)$$
$$+\tau_2(|\beta_\mu|+\beta_\mu)-|\alpha_\lambda|-|\beta_\mu|+L+M]$$
=A

for $R\geq 1$

and

$$|F(z)|\leq|a_n|R^{n+1}+R[|\alpha_n|+|\beta_n|-k_{1(}(|\alpha_n|+\alpha_n)-k_2(|\beta_n|+\beta_n)+\tau_1(|\alpha_\lambda|+\alpha_\lambda)$$
$$+\tau_2(|\beta_\mu|+\beta_\mu)-|\alpha_\lambda|-|\beta_\mu|+L+M]-(1-R)(|\alpha_0|+|\beta_0|)$$
=B

for $R\leq 1$.

Hence , by using Lemma 1, it follows that the number of zeros of F(z) and therefore P(z) in $\dfrac{|a_0|}{X}\leq|z|\leq\dfrac{R}{c}, c>1$ is less than or equal to $\dfrac{1}{\log c}\log\dfrac{A}{|a_0|}$ for $R\geq 1$ and the number of zeros of F(z) and therefore P(z) in $\dfrac{|a_0|}{Y}\leq|z|\leq\dfrac{R}{c}, c>1$ is less than or equal to $\dfrac{1}{\log c}\log\dfrac{B}{|a_0|}$ for $R\leq 1$.

That completes the proof of Theorem 1.

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