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RESEARCH ARTICLE

SIGNED MEASURE AND DECOMPOSITION THEOREMS

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ABSTRACT

In this paper we define σ- algebra of sets and give the definition of positive set and negative set and show that any space can be decomposed into the disjoint union of two sets one is positive set and other is negative set with the help of Hahn-Decomposition theorem. Jorden-Decomposition theorem proves that there exist a unique pair (v_1, v_2) of non negative measures such that at least one of them is finite and $v_1 \perp v_2$ and $\vartheta = v_1 - v_2$.

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INTRODUCTION

Definition

Let $\mathcal{A}$ be any σ-algebra on the set X. ϑ be a set function defined on $\mathcal{A}$ s.t.

1. $-\infty \le \vartheta \le \infty$ and ϑ takes at most one of the values-∞ $and\ \infty$.
2. $\vartheta(\phi) = 0$
3. $\vartheta\ is$ Countably additive, then ϑ is called a Signed Measure on the space (X, $\mathcal{A}$, ϑ).

Example

1. Let μ be any measure and $\vartheta = -\mu$, then ϑ is a signed measure.
2. Let λ and μ be any two measures s.t. at least one of them is finite. Take $\vartheta = \lambda - \mu$. Then ϑ is a signed measure.

Properties

1. ϑ is finitly additive.

Proof

Let $E_1, E_2, \ldots, E_n$ be finitly many disjoint measurable sets. And E=$\bigcup_{i=1}^{n} Ei$, Define $E_i = \phi$ for i> n. Then (Ei) is a disjoint sequence of measurable sets, Hence $\vartheta(\bigcup_{i=1}^{n} Ei) = \sum_{i=1}^{n} \vartheta(Ei)$.

2. If ϑ(E) is finite and F is a measurable subset of E, then $\vartheta(F)$ is finite.

Proof

E = F∪(E-F) ⇒ $\vartheta(E) = \vartheta(F) + \vartheta(E-F)$, Since $\vartheta(E)$ is finite it follows immediately that $\vartheta(F)$ is finite.

3. ϑ is subtractive i.e. if A $\supset B$ be the measurable sets, and $\vartheta(B)$is finite and then $\vartheta(A-B) = \vartheta(A) - \vartheta(B)$.

Proof

A= B∪(A-B) ⇒ $\vartheta(A) = \vartheta(B) + \vartheta(A-B) \Rightarrow \vartheta(A-B) = \vartheta(A) - \vartheta(B)$.

Example

Signed measure may not be monotone.

4. Let $(\Re, \mathcal{M}, m)$ be L^1 measure space and λ is the unit measure concentrated at 0, defined on $\mathcal{M}$.

$$\text{Let } \vartheta = \lambda - m,\ \lambda(E) = \begin{cases} 0\ if\ 0 \notin E \\ 1\ if\ 0 \in E \end{cases}$$

then ϑ is a signed measure. [Because $\lambda\ is$ finite]

Let E= [0,1] and F = (0,1) and G = [0, $\frac{1}{2}$] then F ⊂ E , G ⊂ $E\ and\ \vartheta(F) < \vartheta(E), \vartheta(G) > \vartheta(E)$. Hence ϑ is not monotone.

Let {En} be any sequence of disjoint measurable sets, and then $\sum_{n=1}^{\infty}\vartheta(En)$ is either properly divergent or absolutely convergent.

Proof

E = $\bigcup_1^{\infty} En$ then $\vartheta(E) = \sum_{n=1}^{\infty}\vartheta(En)$.

Case I

Let $\vartheta(E) = \infty$, then $\sum_{n=1}^{\infty}\vartheta(En) = \infty$ □ $\sum_{n=1}^{\infty}\vartheta(En)$ is properly divergent.

Case II

Let $\vartheta(E) = -\infty$, then $\sum_{n=1}^{\infty}\vartheta(En) = -\infty$ □ $\sum_{n=1}^{\infty}\vartheta(En)$ is properly divergent.

Case III

Suppose $\vartheta(E)$ is finite. Let (Fn) be any permutation of (En) , then $\sum_{n=1}^{\infty}\vartheta(Fn)$ is a derangement of $\sum_{n=1}^{\infty}\vartheta(En)$.

Since (Fn) is a disjoint sequence and $\bigcup_1^{\infty} Fn$=$\bigcup_1^{\infty} En$, therefore $\vartheta(\bigcup_1^{\infty} Fn) = \vartheta(\bigcup_1^{\infty} En)$= $\vartheta(E) \Rightarrow \sum_{n=1}^{\infty}\vartheta(Fn) = \vartheta(E)$

Which shows that every derangement of $\sum_{n=1}^{\infty}\vartheta(En)$ is convergent. Hence $\sum_{n=1}^{\infty}\vartheta(En)$ is absolutely convergent.

Definition

Let N be any measurable set such that $\vartheta(M)$ =0 for every measurable subset M of N. Then N is called a null set of ϑ.

Remark

1. A measurable sub set of a null set is a null set.
2. A countable union of null sets is a null set.

Let {N_k} be any sequence of null sets for ϑ, write N = $\bigcup_{n=1}^{\infty} N_k$ and M be any measurable subset of N.

Define $M_1 = N_1$, $M_2 = N_2 - N_1$, M_3= N_3-($N_1 \cup N_2$),…………

Then {M_k} is a disjoint sequence of Null sets, and $\bigcup_{n=1}^{\infty} M_k$= $\bigcup_{n=1}^{\infty} N_k$= N

M □ N ⊔ $M = M \cap N$= M∩ [$\bigcup_{n=1}^{\infty} M_k$] = $\bigcup_{n=1}^{\infty}[M \cap M_k]$⊔ $\vartheta(M)$= $\sum_{k=1}^{\infty}\vartheta([M \cap M_k)$ = 0.   [As M_k is a null set]

Proved.

Definition

Let ϑ be any signed measure and P is a measurable set s.t. $\vartheta(E) \geq 0$ for every measurable set E□ P , then P is called a Positive set for ϑ.

Let Q be any measurable set such that $\vartheta(E) \leq 0$ for every measurable set E of Q , then Q is called a Negative set for ϑ.

Example

Let ($\mathbb{R}, \mathcal{M}, m$) be the L-measure space, λ be any unit measure concentrated at 1. then $\vartheta = \lambda - m$ is a signed measure.

Let P be a countable set of ℝ , Let E be any measurable sub set of P

$$\vartheta(E) = \lambda(E) - m(E) = \lambda(E) = \begin{cases} 0 \; if \; 1 \notin E \\ 1 \; if \; 1 \in E \end{cases}$$

□ $\vartheta(E) \geq 0$ □ E □ P ⇒ P is a positive set for ϑ.

Further let Q be any L'-measurable set s.t. 1 □ Q.

Let E be any measurable sub set of Q then $\vartheta(E) = \lambda(E) - m(E) = 0 - m(E) = -m(E) \leq 0$ [Because 1 □ E] ⇒ Q is a Negative set for ϑ.

Remark

1. Every measurable subset of positive set is a positive set.
2. A Countable union of positive sets is a positive set.

Let (Pn) be any sequence of Positive sets and P=$\bigcup(Pn)$, Define T_1= P_1 , T_2=P_2-P_1, T_3 = P_3-(P_1 □P_2), …….

Then (T_n) is a disjoint sequence of Positive sets and P = $\bigcup_1^{\infty}$Pn = $\bigcup_1^{\infty}$Tn

Let E be any measurable subset of P.

Then E = E∩ P =E∩ ($\bigcup_1^{\infty}$Tn) = $\bigcup_1^{\infty}$(E ∩ Tn) □ $v(E) = \sum_{n=1}^{\infty} v(E \cap Tn) \geq 0$

Shows that P is a positive set for v.

Lemma

Let E be any measurable set with $0 < v(E) < \infty$, then a positive sub set A of E s.t. $0 < v(A) < \infty$.

Proof

If E is a positive set then no further argument is needed.

Suppose E is not a positive set, Then there are two cases.

Case I

Suppose there exist finitly many disjoint measurable subsets E1, E2,……….E_n of E such that $v(E_i) < 0$ and E-$\bigcup_{i=1}^{n} E_i$ is a positive set ,Take $F = \bigcup_{i=1}^{n} E_i$, $E_i \subset E \Rightarrow v(Ei) < \infty$ for $1 \le i \le n \Rightarrow v(F) = \sum_{n=1}^{\infty} v(Ei)$ is finite.

Let P = E-F then P is a positive set, $v(E) is\ finite$ as $P \subset E$ and $v(P) = v(E) - v(F) > v(E)$ [As $v(F) < 0$] $0 < v(E) < v(P) < \infty$ $0 < v(P) < \infty$.

Case II

Suppose that the case (I) does not hold, that there is some E which is not a positive set, that E contains at least one set of negative measure.

Let n_1 be the smallest positive integer such that E contains a measurable set E_1 with $v(E_1) < -\frac{1}{n_1}$

Consider $E - E_1$ then by supposition $E - E_1$ is not a positive set. That means $E - E_1$ contains a set of negative measure. Let n_2 be the smallest positive integer such that $E - E_1$ contains a measurable set E_2 with $v(E_2) < -\frac{1}{n_2}$
Now $E - (E_1 \cup E_2)$ is not a positive set.

By repeating the above argument indefinitely , we obtain a sequence $\{E_n\}$ of disjoint measurable subsets of E and a sequence $(n_k)_{k=1}^{\infty}$ of positive integers such that n_k is the smallest positive integer with the property that $E - \left(\bigcup_{i=1}^{k-1} E_i\right)$ contains a measurable set E_k with $v(E_k) < -\frac{1}{n_k}$

Define A = $E - \left(\bigcup_{k=1}^{\infty} E_k\right)$ as A is a subset of E, we have $v(A) s\ finite.$

By the same argument $v(\bigcup_{k=1}^{\infty} E_k)$ is finite.

Since v is subtractive (under the condition of finiteness)

We have $v(A) = v(E) - v\left(\bigcup_{k=1}^{\infty} E_k\right) = v(E) - \sum_{1}^{\infty} v(E_k) \ge v(E)$

Therefore $0 < v(E) < v(A) < \infty \Rightarrow 0 < v(A) < \infty$. Now to show that A is a positive set.

As $v(E_k) < -\frac{1}{n_k} \Rightarrow \frac{1}{n_k} < -v(E_k) \Rightarrow \frac{1}{n_k} < |v(E_k)|$ and $\sum_{k=I}^{\infty} v(E_k)$ is absolutely convergent. [As $\sum_{k=I}^{\infty} v(E_k) = v(\bigcup_{1}^{\infty} E_k)$ is finite]

Therefore by comparison test $\sum_{k=I}^{\infty} \frac{1}{n_k}$ is convergent

 $\lim_{k \to \infty} \left(\frac{1}{n_k}\right) = 0$, Let $\varepsilon > 0$ chose k such that $\frac{1}{n_k - 1} < \varepsilon$. By the choice of n_k the set $E - \left(\bigcup_{i=1}^{k-1} E_i\right)$ contains no set of measure <- $\frac{1}{n_k - 1}$
 A has no measurable subset of measure <- $\frac{1}{n_k - 1}$
 A has no measurable sub set of measure <- $\varepsilon \Rightarrow v(S) \ge 0$ S A
 A is a positive set and lemma is proved.

Note: Every measurable set of finite positive measure contains a positive set of finite positive measure.

Hahn Decomposition Theorem

Let v be any signed measure then there exist a measurable set P such that P is a positive set and X- P is a negative set for v.

Proof

Assume that v does not take the value ∞

Let α = Sup{ v(A) | A is a positive set for v}. By the property of superemum, there exist a sequence $\{A_n\}$ of positive sets such that $v(A_n) \to \alpha$. Let $P = \bigcup_{1}^{\infty} A_n$, then P is a positive set for v.
To show that X- P is a negative set for v. Before this we have to show that $v(P) = \alpha$.

Since P is a positive set, by definition of α we get $v(P) \leq \alpha$ .... (1)

As $P = \bigcup_1^\infty A_n$ we have $A_n \subset P \ \forall$ n

□ $P = A_n + (P - A_n)$

□ $v(P) = v(A_n) + v(P - A_n) \geq v(A_n)$

□ $v(P) \geq \lim_{n\to\infty} v(A_n) \Rightarrow v(P) \geq \alpha$ .... (2)

From (1) and (2) we have $v(\text{P}) = \alpha$

To show x-P is a negative set for v.

Suppose this is not so. Which means there exist a measurable set for E □ X-P such that $v\ (E) \geq 0$

By first assumption $v(E) < \infty$

Thus $0 < v(E) < \infty$

From the preceding lemma we can find a measurable set A □ E such that A is a positive set and $0 < v(A) < \infty$

Let S = P∪ A, then S is a positive set for v [As P is a positive set]

As P∩ A=ϕ, we get $v(S) = v(P) + v(A) > v(P)$ [Because $v(A) \geq 0$, $v(P)$ is finite]

□ $v(S) > \alpha$ which is a contradiction to the choice of α,

Hence X-P must be a negative set.

Definition

Let v be any signed measure on X. Let P be a measurable set such that P is a positive set and X-P is a negative set for v. Then the pair (P, X-P) is called Hahn Decomposition for v.

Remark

The above theorem shows that every signed measure has a Hahn Decomposition. We now wish to know about the uniqueness. The following example shows that Hahn-Decomposition may not be unique.

Example

Let $(\mathbb{R}, M, m)$ *be* L^1- measurable space, λ be the unit measure on $\mathbb{R}$ concentrated on 1.

Define $v = \lambda - m$,then v is a signed measure on $\mathbb{R}$.

Let P be any countable set of $\mathbb{R}$ such that $1 \in$ P. Let E⊂ P be measurable then

$$v(E) = \lambda(E) - m(E) = \lambda(E) = \begin{cases} 0 \ if \ 1 \notin E \\ 1 \ if \ 1 \in E \end{cases} \qquad \text{[as m(E) = 0]}$$

Shows that P is a positive set for v.

Let F □ $\mathbb{R} - P$ *be measurable* then $v(F) = \lambda(F) - m(F) = -m(F) \leq 0$ [Because 1 □ $\mathbb{R} - P$ *and* F $\subset \mathbb{R} - P \Rightarrow \lambda(F) = 0$]

Shows that $\mathbb{R} - P$ is a negative set for v.

Hence $(P, \mathbb{R} - P)$ is a Hahn-Decomposition for v.

Further Let N be the set natural numbers and $\mathbb{Z}$ be the set of integers. Then it follows that $(N, \mathbb{R} - N)$ is a Hahn-Decomposition and $(\mathbb{Z}, \mathbb{Z} - N)$ is also a Han-Decomposition for v.

However we note that $v(\mathbb{Z} - N) = \lambda((\mathbb{Z} - N) - m(\mathbb{Z} - N) = 0 - 0 = 0 \Rightarrow v\ (\mathbb{Z} - N) = 0$ [$\lambda(\mathbb{Z} - N) = 0$ *as* $1 \notin (\mathbb{Z} - N)$ *and* $m((\mathbb{Z} - N) = 0$ *as* $(\mathbb{Z} - N)$ *is countable*]

Let T□ $(\mathbb{Z} - N)$be measurable then $\lambda(T) = v(T) - m(T) = 0$

By the same argument shows that $\mathbb{Z} - N$ is a null set for v. It is clear that $\mathbb{Z} - N$ is also a null set for v.

The following theorem brings out the fact clearly.

Theorem

Let $(P, X - P)$ be any Hahn-Decomposition for v, (S, X-S) is a Hahn-Decomposition iff P ΔS is a null set.

Proof

Suppose (S, X-S) is a Hahn-Decomposition for v.

Let $E \subset \text{P} - \text{S}$ measurable then $v(E) \geq 0$ and $v(E) \leq 0$ [As E□ P and E□ X − S] □ $v(E) = 0$ □ P-S is a null set.

Similarly we can show that S − P is also a null set.

Hence $P \Delta S$ = (P-S) □ $(S - P)$ is a null set.

Conversely

Suppose that $P \Delta S$ is a null set then P-($P \Delta S$) $= P \cap S$ is measurable.

Also ($P \Delta S$)-P = S-P is measurable □$(P \cap S)$ □ $(S - P)$s measurable $\Rightarrow (P \cap S)$ □ $(S - P) = S$ is measurable.

Let $E \subset S$ be measurable then E= $E \cap X = E \cap (P \cup X - P)$ = $(E \cap P)$ □ $\left(E \cap X - P\right)$ = $(E \cap P)$ □ $(E \cap S - P)$ [Because $E \subset S$]

□ $v(E) = v(E \cap P) + v(E \cap S - P) = v(E \cap P) \geq 0$ [as $E \cap S$ and $S - P$ are null sets]

□ S is a positive set.

To show that X-S is a negative set.

Let $F \subset X - S$ be measurable then $F = F \cap X = F \cap [X - P \cup P] = (F \cap X - P) \cup (F \cap P) = (F \cap X - P) \cup (P - S)$ [Because F □X-S]

□ $v(F) = v(F \cap X - P) + v(P - S)$ [P-S is a null set]

$= v(F \cap X - P) \leq 0$ [X-P is a negative set]

Shows that X-S is a negative set for v, Hence proved.

Note: Above said theorem can also be stated as that Hahn-Decomposition is unique up to null sets.

Definition

Let v be any signed measure and S be any measurable subset of X. if X-S is a null set for v. Then we say that v is supported on S.

Definition

Let v_1, v_2 be two signed measures, then there exist a measurable set S s.t. v_1 is supported on S and v_2 is supported on X-S . Then we say that v_1 is orthogonal or singular to v_2 and we write as $v_1 \perp v_2$.

Remark: $v_1 \perp v_2 \Leftarrow v_2 \perp v_1$ therefore we can say that v_1 *and* v_2 are mutually orthogonal.

Jorden Decomposition Theorem

Statement

Let v be any signed measure, Then there exist a unique pair (v_1 , v_2) of non negative measures such that at least one of them is finite and $v_1 \perp v_2$ and $v = v_1 - v_2$.

Lemma 1

Let v be a signed measure S be any measurable set and v'(E)= $v(E \cap S)$ for every measurable set E. Then v' is a signed measure and v' is supported on S.

Proof

It is clear that v' is a signed measure.

To show that it is supported on S, Take E□X-S be measurable then v'(E)= $v(E \cap S)$= $v(\phi)$=0

Which shows that v' is is supported on S.

Lemma 2

Let v be any signed measure on X. S be any measurable sub set of X then for any measurable set E Define $v_1(E) = v(E \cap S)$ and $v_2(E) = -v(E \cap X - S)$ then

1. v_1 *and* v_2 are signed measures.
2. v_1 is supported on S and v_2 is supported on X-S and $v_1 \perp v_2$.
3. $v = v_1 - v_2$.
4. *If* $(S, X{-}S)$ is a Hahn-Decomposition for v then v_1 *and* v_2 are both non negative and atleast one of them is finite.

Proof

(1) and (2) are follows from Lemma (1).

(3)*Let* $E \subset X$ be measurable then E=E∩X= $E \cap (S \cup X - S) = (E \cap S) \cup (E \cap \cap X - S)$

□ $v(E) = v(E \cap S) + v(E \cap \cap X - S) = v_1(E) - v_2(E) = (v_1 - v_2)(E)$

Shows that $v = v_1 - v_2$.

(4) Let $(S, X{-}S)$ is a Hahn-Decomposition for v, E□X be measurable then

$v_1(E) = v(E \cap S) \geq 0$ [Because S is a positive set]

$v_2(E) = -v(E \cap \bar{X - S}) \geq 0$ [Because X-S is a negative set]

Shows that v_1 and v_2 are both non negative.

Further if v does not take the value $+\infty$ then v_1 is finite .and if v does not take the value $+\infty$ then v_2 is finite. This proves the Lemma.

Lemma 3

Converse of the Lemma (2)

Let v_1 and v_2 are both non negative measures. Then atleast one of them is finite and $v = v_1 - v_2$. Then

1. v is a signed measure.
2. $If\ v_1 \perp v_2$ then v_1 is supported on S and v_2 is supported on X-S.
3. Then $(S, X{-}S)$ is a Hahn-Decomposition for v and $v_1(E) = v(E \cap S)$ and $v_2(E) = -v(E \cap \bar{X - S})$ $\forall$ measurable set E.

Proof

(1) is obvious.

(2) Let v_1 is supported on S and v_2 is supported on X-S.

E□ S be measurable. Then $v(E) = v_1(E) - v_2(E) = v_1(E) \geq 0$ [Bacause $v_2 = 0$ on S]

Shows that S is a positive set for v.

Let F □ X − S be measurable then $v(F) = v_1(F) - v_2(F) = -v_2(F) \leq 0$ [$v_2 \geq 0, v_1 = 0$ on X-S]

Shows that X-S is negative set for v.

(3) Finally Let E⊂ X be measurable then $v_1(E) = v_1(E \cap S) + v_1\left(E \cap \bar{X - S}\right)$

$= v_1(E \cap S) = v_1(E \cap S) - v_2(E \cap S) = v(E \cap S)$ [$v_1 = 0$ on X-S and $v_2 = 0$ on S]

In the same manner $v_2(E) = -v\left(E \cap \bar{X - s}\right)$.

Proof of the theorem

Let (P, X-P) be a Hahn-Decomposition for v.

For a measurable set E define $v_1 = v(E \cap P)$ and $v_2 = -v\left(E \cap \bar{X - P}\right)$.

Then v_1 and v_2 are signed measures and $v_1 \perp v_2$, and atleast one of them is finite and $v = v_1 - v_2$.

Which proves the existence part of the theorem.

Uniqueness

Suppose that (v_1', v_2') be another pair of non negative measures such that $v_1' \perp v_2'$ and at least one of them is finite and $v = v_1' - v_2'$. Since $v_1' \perp v_2'$

There exist a measurable set S such that (S, X-S) is a Hahn-Decomposition for $v = v_1' - v_2'$.

As Hahn-Decomposition is unique up to null sets we have P Δ S is a null set.

Let E be any measurable set.

Then by Lemma (3) $v_1' = v(E \cap S) = v[(E \cap S \cap P) \cup (E \cap S \cap P^c)] = v[(E \cap S \cap P)] + v[(E \cap (S \cap P^c)] = v[(E \cap S \cap P)] + v\left[\left(E \cap \bar{S - P}\right)\right] = v[(E \cap S \cap P)]$ [Because P Δ S is a null set] …………….(1)

Also $v_1(E) = v(E \cap P) = v[(E \cap S \cap P) \cup (E \cap P \cap S^c)] = v[(E \cap S \cap P)] + v\left[\left(E \cap \bar{P - S}\right)\right] = v[(E \cap S \cap P)]$

[Because P Δ S is a null set] ……………(2)

From (1) and (2) we get $v_1'(E) = v_1(E)$ ∀ E □ X

In the same manner $v_2'(E) = v_2(E)$ ∀ E □ X □ $v_1' = v_1$ and $v_2' = v_2$ which completes the proof.

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